J. Math. Anal. Appl. 344 (2008) 1120–1142

Contents lists available at ScienceDirect

J. Math. Anal. Appl.

www.elsevier.com/locate/jmaa

Regularity results for a class of obstacle problems under nonstandard growth conditions

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ARTICLE INFO

Article history:
Received 11 February 2008
Available online 3 April 2008
Submitted by V. Radulescu

Keywords:
Obstacle problems
Nonstandard growth
Minimizers of functionals
Hölder continuity

ABSTRACT

We prove regularity results for minimizers of functionals $\mathcal{F}(u,\Omega) := \int_\Omega f(x,u,Du)\,dx$ in the class $K := \{u \in W^{1,p(x)}(\Omega,\mathbb{R})\colon\ u \geqslant \psi\}$, where $\psi : \Omega \to \mathbb{R}$ is a fixed function and f is quasiconvex and fulfills a growth condition of the type

$$L^{-1}|z|^{p(x)} \leqslant f(x,\xi,z) \leqslant L\big(1+|z|^{p(x)}\big),$$

with growth exponent $p : \Omega \to (1,\infty)$.

1. Introduction

The aim of this paper is to study the regularity properties for local minimizers of integral functionals of the type

$$\mathcal{F}(u,\Omega) := \int_\Omega f\big(x,u(x),Du(x)\big)\,dx, \tag{1.1}$$

in the class $K := \{u \in W^{1,p(x)}(\Omega;\mathbb{R})\colon\ u \geqslant \psi\}$, where ψ is a fixed obstacle function, Ω is a bounded open set in $\mathbb{R}^n$ and $f : \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ is a Carathéodory function satisfying a growth condition of the type

$$L^{-1}|z|^{p(x)} \leqslant f(x,\xi,z) \leqslant L\big(1+|z|^{p(x)}\big), \tag{1.2}$$

for all $x \in \Omega$, $\xi \in \mathbb{R}$, $z \in \mathbb{R}^n$, with $p : \Omega \to (1,+\infty)$ a continuous function and $L \geqslant 1$. We do not assume the functional considered in (1.1) to admit an Euler–Lagrange equation, especially not the integrand to be twice differentiable. Our assumptions on the integrand f are quasiconvexity (see (H2)) and $p(x)$ growth in the sense of (H1).

Problems with nonstandard growth became of increasing interest in the past ten years, on one hand since they appear for example in a natural way in the modeling of non-newtonian fluids (for example electrorheological fluids, see for instance [3,29]), on the other hand since they are in particular interesting from the mathematical point of view, representing the borderline case between standard growth and so-called (p,q) growth conditions (see for example [12] for regularity results in the case of (p,q) grwoth).

It is not difficult to see that existence of local minimizers for problems of $p(x)$ type under typical structure conditions can basically be shown in the generalized Sobolev space $W^{1,p(x)}_{\mathrm{loc}}(\Omega)$ (see Definition 2.1 for more details). These spaces can be interesting by themselves. So there have been made a lot of investigations on their properties, see for example [8,10,11,22,23,29].

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0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2008.03.068

Mathematical investigations of regularity for problems with $p(x)$ growth started with a first higher integrability result of Zhikov [30] for functionals of a special type. Then Acerbi and Mingione [1,2] showed $\mathcal{C}^{0,\alpha}$ regularity for minimizers of functionals $\int f(x, Du)\,dx$ under certain weak continuity assumptions on the exponent function p. Coscia and Mingione [7] were able to show that, in order to obtain $\mathcal{C}^{1,\alpha}$ regularity, one needs Hölder continuity of the exponent function p itself. The authors (see [14,25]) were able to extend results of this type to functionals $\int f(x, u, Du)\,dx$ and to higher order functionals $\int f(x, u, Du, \dots, D^m u)\,dx$ with $p(x)$ growth. All of these papers make use of the so-called "blow up technique" in their proofs. Recently, regularity results of this type were also shown by the method of A-harmonic approximation by Zatorska-Goldstein and one of the authors [27].

In this paper we are concerned with one sided obstacle problems with $p(x)$ growth, providing regularity results in the setting of Hölder and Morrey spaces. Obstacle problems of this type in the situation of standard growth $p = \text{const.}$ have been studied by Choe [5], where regularity in Morrey spaces was considered, and by one of the authors [15], where these results have been extended in a sharp way. It turns out that the results of [15] can be used for our purposes, providing adequate reference estimates.

To the knowledge of the authors, the present paper seems to be a first regularity result for obstacle problems with $p(x)$ growth.

The techniques in this paper are a combination of those in [15], providing the reference estimates, and suitable localization and freezing techniques to treat the nonstandard growth exponent. The regularity assumptions for the exponent function p enable us to establish appropriate comparison estimates between the original minimizer and the minimizer of the frozen problem.

In the first part of the paper (see Theorem 2.8) we show $\mathcal{C}^{0,\alpha}$ regularity for minimizers of functionals of the type $\int f(x, Du)\,dx$ in the case where the exponent function satisfies a weak regularity condition in the sense of (2.8) and the obstacle lies in an appropriate Morrey space. In Theorem 2.9, we extend these results to the case of more general functionals $\int f(x, u, Du)\,dx$. Therefore we take use of the so-called Ekeland variational principle, a tool that revealed to be crucial in regularity since the paper [18]. Finally, in Theorem 2.10 we prove $\mathcal{C}^{1,\beta}$ regularity of minimizers in the case that the function p is $\mathcal{C}^{0,\alpha}$ and the obstacle lies in an appropriate Campanato space which is isomorphic to some Hölder space.

The results of this paper could be used to prove estimates of Calderón–Zygmund type (as done by Acerbi and Mingione for equations in [4] and extended to systems of higher order by one of the authors in [26]) also for obstacle problems with $p(x)$ growth.

2. Notation and statements

In the sequel Ω will denote an open bounded domain in $\mathbb{R}^n$ and $B(x, R)$ the open ball $\{y \in \mathbb{R}^n\colon\ |x - y| < R\}$. If u is an integrable function defined on $B(x, R)$, we will set

$$(u)_{x,R} = \fint_{B(x,R)} u(x)\,dx = \frac{1}{\omega_n R^n} \int_{B(x,R)} u(x)\,dx,$$

where ω_n is the Lebesgue measure of $B(0, 1)$. We shall also adopt the convention of writing B_R and $(u)_R$ instead of $B(x, R)$ and $(u)_{x,R}$ respectively, when the center will not be relevant or it is clear from the context; moreover, unless otherwise stated, all balls considered will have the same center. Finally the letter c will freely denote a constant, not necessarily the same in any two occurrences, while only the relevant dependences will be highlighted.

We start with the following definition.

Definition 2.1. A function u is said to belong to the generalized Sobolev space $W^{1,p(x)}(\Omega; \mathbb{R})$ if $u \in L^{p(x)}(\Omega; \mathbb{R})$ and the distributional gradient $Du \in L^{p(x)}(\Omega; \mathbb{R}^n)$. Here the generalized Lebesgue space $L^{p(x)}(\Omega; \mathbb{R})$ is defined as the space of measurable functions $f : \Omega \to \mathbb{R}$ such that

$$\int_\Omega \left|f(x)\right|^{p(x)}\,dx < \infty.$$

This is a Banach space equipped with the Luxemburg norm

$$\|f\|_{L^{p(x)}(\Omega;\mathbb{R})} = \inf\left\{\lambda > 0\colon\ \int_\Omega \left|\frac{f(x)}{\lambda}\right|^{p(x)} dx \leqslant 1\right\}.$$

This definition can be extended in a straightforward way to the case of vector-valued functions.

Next, we will set

$$\mathcal{F}(u, \mathcal{A}) := \int_{\mathcal{A}} f\big(x, u(x), Du(x)\big)\,dx,$$

for all $u \in W^{1,1}_{\mathrm{loc}}(\Omega)$ and for all $\mathcal{A} \subset \Omega$.

We adopt the following notion of local minimizer.

Definition 2.2. We say that a function $u \in W^{1,1}_{\mathrm{loc}}(\Omega)$ is a local minimizer of the functional (1.1) if $|Du(x)|^{p(x)} \in L^1_{\mathrm{loc}}(\Omega)$ and

$$\int_{\operatorname{spt}\varphi} f\big(x, u(x), Du(x)\big)\,dx \leqslant \int_{\operatorname{spt}\varphi} f\big(x, u(x)+\varphi(x), Du(x)+D\varphi(x)\big)\,dx,$$

for all $\varphi \in W^{1,1}_0(\Omega)$ with compact support in Ω.

We shall consider the following growth, ellipticity and continuity conditions:

$$L^{-1}\big(\mu^2+|z|^2\big)^{p(x)/2} \leqslant f(x,\xi,z) \leqslant L\big(\mu^2+|z|^2\big)^{p(x)/2}, \tag{H1}$$

$$\int_{Q_1}\big[f\big(x_0,\xi_0,z_0+D\varphi(x)\big)-f(x_0,\xi_0,z_0)\big]\,dx \geqslant L^{-1}\int_{Q_1}\big(\mu^2+|z_0|^2+\big|D\varphi(x)\big|^2\big)^{\frac{p(x_0)-2}{2}}\big|D\varphi(x)\big|^2\,dx \tag{H2}$$

for some $0 \leqslant \mu \leqslant 1$, for all $z_0 \in \mathbb{R}^n$, $\xi_0 \in \mathbb{R}$, $x_0 \in \Omega$, $\varphi \in C^\infty_0(Q_1)$, where $Q_1 = (0,1)^n$,

$$\big|f(x,\xi,z)-f(x_0,\xi,z)\big| \leqslant L\omega_1\big(|x-x_0|\big)\big[\big(\mu^2+|z|^2\big)^{p(x)/2}+\big(\mu^2+|z|^2\big)^{p(x_0)/2}\big]\big[1+\big|\log\big(\mu^2+|z|^2\big)\big|\big] \tag{H3}$$

for all $z \in \mathbb{R}^n$, $\xi \in \mathbb{R}$, x and $x_0 \in \Omega$, where $L \geqslant 1$. Here $\omega_1 : \mathbb{R}^+ \to \mathbb{R}^+$ is a nondecreasing continuous function, vanishing at zero, which represents the modulus of continuity of p:

$$\big|p(x)-p(y)\big| \leqslant \omega_1\big(|x-y|\big). \tag{H4}$$

Throughout this paper we will assume that ω_1 satisfies the condition

$$\limsup_{R\to 0}\omega_1(R)\log\left(\frac{1}{R}\right) < +\infty, \tag{2.1}$$

thus in particular, without loss of generality, we may assume that

$$\omega_1(R) \leqslant L|\log R|^{-1}, \tag{2.2}$$

for all $R < 1$.

We shall also consider the following continuity condition with respect to the second variable:

$$\big|f(x,\xi,z)-f(x,\xi_0,z)\big| \leqslant L\omega_2\big(|\xi-\xi_0|\big)\big(\mu^2+|z|^2\big)^{p(x)/2}, \tag{H5}$$

for any $\xi, \xi_0 \in \mathbb{R}$. Without loss of generality, we shall suppose that ω_2 is a concave, bounded and, hence, subadditive function. We note that no differentiability is assumed on f with respect to x or with respect to z.

Since all our results are local in nature, without loss of generality we shall suppose that there exist $\gamma_1, \gamma_2 \in (1,+\infty)$, $\gamma_1 \leqslant \gamma_2$ such that

$$1 < \gamma_1 \leqslant p(x) \leqslant \gamma_2 \quad \text{for all } x \in \Omega, \tag{2.3}$$

and

$$\int_\Omega \big|Du(x)\big|^{p(x)}\,dx < +\infty. \tag{2.4}$$

Finally we set

$$K := \big\{u \in W^{1,p(x)}(\Omega;\mathbb{R}):\ u \geqslant \psi\big\}, \tag{2.5}$$

where $\psi \in W^{1,p(x)}(\Omega;\mathbb{R})$ is a fixed function.

Let us recall the definition of Morrey and Campanato spaces (see for example [21]).

Definition 2.3 *(Morrey spaces).* Let Ω be an open and bounded subset of $\mathbb{R}^n$, let $1 \leqslant p < +\infty$ and $\lambda \geqslant 0$. By $L^{p,\lambda}(\Omega)$ we denote the linear space of functions $u \in L^p(\Omega)$ such that

$$\|u\|_{L^{p,\lambda}(\Omega)} := \left\{\sup_{x_0\in\Omega,\,0<\rho<\operatorname{diam}(\Omega)} \rho^{-\lambda}\int_{\Omega(x_0,\rho)}\big|u(x)\big|^p\,dx\right\}^{1/p} < +\infty,$$

where we set $\Omega(x_0,\rho) := \Omega \cap B(x_0,\rho)$.

It is not difficult to see that $\|u\|_{L^{p,\lambda}(\Omega)}$ is a norm respect to which $L^{p,\lambda}(\Omega)$ is a Banach space.

Definition 2.4 *(Campanato spaces).* Let Ω be an open and bounded subset of $\mathbb{R}^n$, let $p \geqslant 1$ and $\lambda \geqslant 0$. By $\mathcal{L}^{p,\lambda}(\Omega)$ we denote the linear space of functions $u \in L^p(\Omega)$ such that

$$[u]_{p,\lambda} := \left\{ \sup_{x_0\in\Omega,\, 0<\rho<\operatorname{diam}(\Omega)} \rho^{-\lambda} \int_{\Omega(x_0,\rho)} \left|u(x) - (u)_{x_0,\rho}\right|^p dx \right\}^{1/p} < +\infty,$$

where $\Omega(x_0,\rho) := \Omega \cap B(x_0,\rho)$ and

$$(u)_{x_0,\rho} := \frac{1}{|\Omega(x_0,\rho)|} \int_{\Omega(x_0,\rho)} u(x)\,dx$$

is the average of u in $\Omega(x_0,\rho)$. Also in this case it is not difficult to show that $\mathcal{L}^{p,\lambda}(\Omega)$ is a Banach space equipped with the norm

$$\|u\|_{\mathcal{L}^{p,\lambda}(\Omega)} := \|u\|_{L^p(\Omega)} + [u]_{p,\lambda}.$$

Remark 2.5. The local variants $L^{p,\lambda}_{\mathrm{loc}}(\Omega)$ and $\mathcal{L}^{p,\lambda}_{\mathrm{loc}}(\Omega)$ are defined in a standard way:

$$u \in L^{p,\lambda}_{\mathrm{loc}}(\Omega) \quad \Leftrightarrow \quad u \in L^{p,\lambda}(\Omega') \quad \forall \Omega' \Subset \Omega,$$
$$u \in \mathcal{L}^{p,\lambda}_{\mathrm{loc}}(\Omega) \quad \Leftrightarrow \quad u \in \mathcal{L}^{p,\lambda}(\Omega') \quad \forall \Omega' \Subset \Omega.$$

The interest of Campanato's spaces lies mainly in the following result which will be used in the next sections.

Theorem 2.6. *Let Ω be a bounded open Lipschitz domain of $\mathbb{R}^n$, and let $n < \lambda < n + p$. Then the space $\mathcal{L}^{p,\lambda}(\Omega)$ is isomorphic to $C^{0,\alpha}(\bar{\Omega})$ with $\alpha = \frac{\lambda-n}{p}$. We also remark that, using Poincaré's inequality, we have that, for a weakly differentiable function v, if $Dv \in L^{p,\lambda}(\Omega)$, then $v \in \mathcal{L}^{p,p+\lambda}(\Omega)$.*

Remark 2.7. Theorem 2.6 also holds for a larger class of domains (see [21, Section 2.3]).

The first result we are able to obtain is for local minimizers in K of the functional

$$\mathcal{H}(u,\Omega) = \int_\Omega h\big(x, Du(x)\big)\,dx, \tag{2.6}$$

where $h : \Omega \times \mathbb{R}^n \to \mathbb{R}$ is a continuous function fulfilling growth, ellipticity and continuity conditions of the kind (H1)–(H3). More precisely we have

Theorem 2.8. *Let $u \in W^{1,1}_{\mathrm{loc}}(\Omega)$ be a local minimizer of the functional* (2.6) *in the class K, defined in* (2.5)*, where h is a continuous function satisfying* (H1)–(H4)*; suppose moreover that the function ψ fulfills the assumption*

$$D\psi \in L^{q,\lambda}_{\mathrm{loc}}(\Omega), \tag{2.7}$$

for some $n - \gamma_1 < \lambda < n$, with $q = \gamma_2 r$ for some $r > 1$, where γ_1 and γ_2 have been introduced in (2.3)*. Finally assume that*

$$\lim_{R\to 0} \omega_1(R) \log\left(\frac{1}{R}\right) = 0. \tag{2.8}$$

Then $u \in C^{0,\alpha}_{\mathrm{loc}}(\Omega)$ with $\alpha = 1 - \frac{n-\lambda}{\gamma_1}$.

The main result of this paper is concerned with local minimizers of the functional (1.1) in K.

Theorem 2.9. *Let $u \in W^{1,1}_{\mathrm{loc}}(\Omega)$ be a local minimizer of the functional* (1.1) *in the class K, defined in* (2.5)*, where f is a continuous function satisfying* (H1)–(H5)*; suppose moreover that the function ψ fulfills* (2.7)*, for some $n - \gamma_1 < \lambda < n$, with $q = \gamma_2 r$ for some $r > 1$, where γ_1 and γ_2 have been introduced in* (2.3)*. Finally assume that*

$$\lim_{R\to 0} \omega_1(R) \log\left(\frac{1}{R}\right) + \omega_2(R) = 0. \tag{2.9}$$

Then $u \in C^{0,\alpha}_{\mathrm{loc}}(\Omega)$ with $\alpha = 1 - \frac{n-\lambda}{\gamma_1}$.

Finally, if the Lagrangian f is more regular and the obstacle stays in an appropriate Campanato space, we have the following result.

Theorem 2.10. *Let $u \in W^{1,1}_{\mathrm{loc}}(\Omega)$ be a local minimizer of the functional* (1.1) *in the class K, defined in* (2.5)*, where f is a function of class $\mathcal{C}^2$ satisfying* (H1)–(H5) *and the function ψ fulfills the assumption*

$$D\psi \in \mathcal{L}^{\gamma_1,\lambda}_{\mathrm{loc}}(\Omega), \tag{2.10}$$

for some $n < \lambda < n + \gamma_1$, where γ_1 has been introduced in (2.3)*. If we assume that*

$$\omega_1(R) + \omega_2(R) \leqslant LR^{\varsigma}, \tag{2.11}$$

for some $0 < \varsigma \leqslant 1$ and all $R \leqslant 1$, then $Du \in \mathcal{L}^{\gamma_1,\tilde{\lambda}}_{\mathrm{loc}}(\Omega)$ for some suitable $n < \tilde{\lambda} < n+\gamma_1$ and therefore $u \in \mathcal{C}^{1,\tilde{\alpha}}_{\mathrm{loc}}(\Omega)$ with $\tilde{\alpha} = 1 - \frac{n-\tilde{\lambda}}{\gamma_1}$.

3. Preliminary results

Before proving our main theorems, we collect some preliminary results and establish some basic notation.

A priori Hölder continuity

We start our preliminary results by quoting a priori Hölder continuity of minimizers of the functional (1.1). We will need that result later in the proof of Theorem 2.9. The proof of the following lemma can be found in [16].

Lemma 3.1. *Let $u \in W^{1,p(x)}(\Omega)$ be a local minimizer of the functional* (1.1) *in the class K, defined in* (2.5)*, where $\psi \in W^{1,1}_{\mathrm{loc}}(\Omega)$ is a given obstacle function fulfilling*

$$D\psi \in L^{q,\lambda}_{\mathrm{loc}}(\Omega), \tag{3.1}$$

with $q = \gamma_2 \tilde{q}$ for some $\tilde{q} > 1$ and $n - \gamma_1 < \lambda < n$, where γ_1 and γ_2 have been introduced in (2.3)*. Suppose moreover that the Lagrangian f satisfies the growth condition* (H1) *and the function p fulfills assumptions* (H4) *and* (2.1)*. Then $u \in \mathcal{C}^{0,\gamma}_{\mathrm{loc}}(\Omega)$ for some $\gamma \in (0,1)$.*

A higher integrability result

We prove a higher integrability result for functionals of the type (1.1).

Lemma 3.2. *Let $\mathcal{O}$ be an open subset of Ω, let $u \in W^{1,1}_{\mathrm{loc}}(\mathcal{O})$ be a local minimizer in K of the functional* (1.1) *with $f : \mathcal{O} \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ satisfying* (H1)*, with the exponent function p satisfying* (H4) *and* (2.1) *and with ψ fulfilling condition* (2.7)*. Moreover suppose that*

$$\int_{\mathcal{O}} \left|Du(x)\right|^{p(x)} dx \leqslant M_1,$$

for some constant M_1. Then, there exist two positive constants c_0, δ depending on n, r, γ_1, γ_2, L, M_1, where r is the quantity appearing in condition (2.7)*, such that, if $B_R \Subset \mathcal{O}$, then*

$$\left(\fint_{B_{R/2}} \left|Du(x)\right|^{p(x)(1+\delta)} dx \right)^{1/(1+\delta)} \leqslant c_0 \fint_{B_R} \left|Du(x)\right|^{p(x)} dx + c_0 \left(\fint_{B_R} \left(\left|D\psi(x)\right|^{p(x)(1+\delta)} + 1 \right) dx \right)^{1/(1+\delta)}. \tag{3.2}$$

Remark. Note that the above higher integrability result and estimate (3.2) also hold for any exponent $\tilde{\delta}$ with $0 < \tilde{\delta} \leqslant \delta$.

Proof of Lemma 3.2. *First step.* We set

$$p_1 := \min_{x \in \overline{B}_R} p(x), \qquad p_2 := \max_{x \in \overline{B}_R} p(x),$$

let $R/2 \leqslant t < s \leqslant R \leqslant 1$, and let $\eta \in \mathcal{C}^\infty_0(B_R)$ be a cut-off function such that $0 \leqslant \eta \leqslant 1$, $\eta \equiv 0$ outside B_s, $\eta \equiv 1$ on B_t, $|D\eta| \leqslant 2(s-t)^{-1}$. Moreover we set $\varphi(x) = \eta(x)(u(x) - (u)_R)$ and let $g = u - \varphi$. We remark that $g = u$ on ∂B_s while on B_t we have $g = (u)_R$, consequently $Dg = 0$ on B_t. We would like to use the minimality of u in K. A priori g is not an element of K, so we set $\tilde{g} := \max\{g, \psi\}$ and $\Sigma := \{x \in \mathbb{R}^n\colon\ g(x) \geqslant \psi(x)\}$. This assures that $\tilde{g} \in K$ and so, by the minimality of u,

$$\mathcal{F}(u, B_s) \leqslant \mathcal{F}(\tilde{g}, B_s). \tag{3.3}$$

Therefore we estimate by (3.3) and the growth (H1)

$$\begin{aligned}
\int_{B_t} \left|Du(x)\right|^{p(x)} dx &\leqslant L \int_{B_s} f\big(x, u(x), Du(x)\big)\, dx \\
&\overset{(3.3)}{\leqslant} L \int_{B_s} f\big(x, \tilde{g}(x), D\tilde{g}(x)\big)\, dx \\
&= L\mathcal{F}(\tilde{g}, B_s \cap \Sigma) + L\mathcal{F}(\tilde{g}, B_s \setminus \Sigma) \\
&= L\mathcal{F}(g, B_s \cap \Sigma) + L\mathcal{F}(\psi, B_s \setminus \Sigma) \\
&\leqslant L \int_{B_s} f\big(x, g(x), Dg(x)\big)\, dx + L \int_{B_s} f\big(x, \psi(x), D\psi(x)\big)\, dx \\
&\overset{(H1)}{\leqslant} L^2 \int_{B_s} \big(1 + \left|Dg(x)\right|^{p(x)}\big)\, dx + L^2 \int_{B_s} \big(1 + \left|D\psi(x)\right|^{p(x)}\big)\, dx \\
&\leqslant L^2 \int_{B_s \setminus B_t} \big[\big(1 - \eta(x)\big)\left|Du(x)\right| + \left|u(x) - (u)_R\right|\left|D\eta(x)\right|\big]^{p(x)} dx + \bar{c} \\
&\leqslant \hat{c} \int_{B_s \setminus B_t} \left|Du(x)\right|^{p(x)} dx + \tilde{c} \int_{B_s} \left|\frac{u(x) - (u)_R}{s - t}\right|^{p(x)} dx + \bar{c} \\
&\leqslant \hat{c} \int_{B_s \setminus B_t} \left|Du(x)\right|^{p(x)} dx + \tilde{c} \frac{1}{|s - t|^{p_2}} \int_{B_R} \left|u(x) - (u)_R\right|^{p(x)} dx + \bar{c},
\end{aligned}$$

where $\hat{c} = L^2 2^{\gamma_2 - 1}$, $\tilde{c} = L^2 2^{2\gamma_2 - 1}$, $\bar{c} = L^2 \int_{B_R} (1 + |D\psi(x)|^{p(x)})\, dx$. Now we proceed in a standard way, 'filling the hole' and applying [21, Lemma 6.1] to deduce

$$\begin{aligned}
\int_{B_{R/2}} \left|Du(x)\right|^{p(x)} dx &\leqslant cR^{p_1 - p_2} \int_{B_R} \left|\frac{u(x) - (u)_R}{R}\right|^{p(x)} dx + c \int_{B_R} \big(1 + \left|D\psi(x)\right|^{p(x)}\big)\, dx \\
&\leqslant cR^{-\omega_1(8R)} \int_{B_R} \left|\frac{u(x) - (u)_R}{R}\right|^{p(x)} dx + c \int_{B_R} \big(1 + \left|D\psi(x)\right|^{p(x)}\big)\, dx \\
&\leqslant c \int_{B_R} \left|\frac{u(x) - (u)_R}{R}\right|^{p(x)} dx + c \int_{B_R} \big(1 + \left|D\psi(x)\right|^{p(x)}\big)\, dx,
\end{aligned}$$

where we used (2.2) and c is a constant depending only on γ_1, γ_2, L. According to the previous facts, we find that

$$\fint_{B_{R/2}} \left|Du(x)\right|^{p(x)} dx \leqslant c \fint_{B_R} \left|\frac{u(x) - (u)_R}{R}\right|^{p(x)} dx + c \fint_{B_R} \big(1 + \left|D\psi(x)\right|^{p(x)}\big)\, dx. \tag{3.4}$$

Second step. Fixing $\vartheta = \min\{\sqrt{\frac{n+1}{n}}, \gamma_1\}$ and taking $R < R_0/16$, where R_0 is small enough to have $\omega_1(8R_0) \leqslant \vartheta - 1$ and therefore $1 \leqslant p_2/p_1 \vartheta \leqslant \vartheta^2 \leqslant (n+1)/n$, by Sobolev–Poincaré's inequality we obtain

$$\begin{aligned}
\fint_{B_R} \left|\frac{u(x) - (u)_R}{R}\right|^{p(x)} dx &\leqslant 1 + \fint_{B_R} \left|\frac{u(x) - (u)_R}{R}\right|^{p_2} dx \\
&\leqslant 1 + c\bigg(\int_{B_R} \big(1 + \left|Du(x)\right|^{p(x)}\big)\, dx\bigg)^{\frac{(p_2 - p_1)\vartheta}{p_1}} R^{\frac{-(p_2 - p_1)\vartheta n}{p_1}} \bigg(\fint_{B_R} \left|Du(x)\right|^{\frac{p_1}{\vartheta}} dx\bigg)^{\vartheta} \\
&\leqslant c(M_1)\bigg(\fint_{B_R} \left|Du(x)\right|^{\frac{p_1}{\vartheta}} dx\bigg)^{\vartheta} + c,
\end{aligned}$$

where in the second inequality we use the fact that $\frac{p_1}{\vartheta} \leqslant \frac{p(x)}{\vartheta} \leqslant p(x)$ and in the last one we use again the fact that, by (2.2), $R^{\frac{-(p_2-p_1)\vartheta n}{p_1}}$ is bounded. So we conclude that

$$\fint_{B_R} \left| \frac{u(x)-(u)_R}{R} \right|^{p(x)} dx \leqslant c \left(\fint_{B_R} \left| Du(x) \right|^{\frac{p(x)}{\vartheta}} dx \right)^{\vartheta} + c. \tag{3.5}$$

Third step. From (3.4) and (3.5) we obtain, for all $R < R_0/16$,

$$\fint_{B_{R/2}} \left| Du(x) \right|^{p(x)} dx \leqslant c_1 \left(\fint_{B_R} \left| Du(x) \right|^{\frac{p(x)}{\vartheta}} dx \right)^{\vartheta} + c_2 \int_{B_R} \left(1 + \left| D\psi(x) \right|^{p(x)}\right) dx,$$

where $c_1 \equiv c_1(\gamma_1, \gamma_2, L, M_1, n)$ and $c_2 \equiv c_2(\gamma_1, \gamma_2, L)$. We now apply Gehring's lemma (see [21, Theorem 6.6] or [19, Chapter V]) to deduce that there exists $0 < \delta < r - 1$ (where r appears in the higher integrability assumption (2.7) on the obstacle function ψ) such that

$$\left(\fint_{B_{R/2}} \left| Du(x) \right|^{p(x)(1+\delta)} dx \right)^{1/(1+\delta)} \leqslant c_0 \fint_{B_R} \left| Du(x) \right|^{p(x)} dx + c_0 \left(\fint_{B_R} \left(\left| D\psi(x) \right|^{p(x)(1+\delta)} + 1 \right) dx \right)^{1/(1+\delta)},$$

with $c_0 \equiv c_0(\gamma_1, \gamma_2, L, M_1, n, r)$. This concludes the proof. $\square$

A remark about local minimizers with obstacles

Let $g : \mathbb{R}^n \to \mathbb{R}$ be a function of class $\mathcal{C}^2$ satisfying, for some $1 < \gamma_1 \leqslant p \leqslant \gamma_2$, the following growth and ellipticity conditions:

$$L^{-1}\left(\mu^2 + |z|^2\right)^{p/2} \leqslant g(z) \leqslant L\left(\mu^2 + |z|^2\right)^{p/2}, \tag{H6}$$

$$\int_{Q_1} \left[g\left(z_0 + D\phi(x)\right) - g(z_0) \right] dx \geqslant L^{-1} \int_{Q_1} \left(\mu^2 + |z_0|^2 + \left| D\phi(x) \right|^2\right)^{(p-2)/2} \left| D\phi(x) \right|^2 dx, \tag{H7}$$

for some $0 \leqslant \mu \leqslant 1$, for all $z_0 \in \mathbb{R}^n$, $\phi \in \mathcal{C}_0^\infty(Q_1)$, where $Q_1 = (0,1)^n$, $L \geqslant 1$. Moreover let v be a local minimizer in the class K of the functional

$$w \mapsto \int_{B_R} g\left(Dw(x)\right) dx, \tag{3.6}$$

with $B_R \Subset \Omega$.

Then it is possible to prove that

$$\int_{B_R} \left\langle A\left(Dv(x)\right), D\varphi(x) \right\rangle dx \geqslant 0, \quad \text{for all } \varphi \in \mathcal{C}_0^\infty(\Omega) \text{ such that } \varphi \geqslant 0, \tag{3.7}$$

where $A(z) := Dg(z)$ and $A(z)$ satisfies the following monotonicity and growth conditions:

$$\left\langle A(z), z \right\rangle \geqslant \nu_1 |z|^p - c, \tag{3.8}$$

for some $\nu_1 \equiv \nu_1(\gamma_1, \gamma_2, L)$ and $c \equiv c(\gamma_1, \gamma_2, L)$, and

$$\left| A(z) \right| \leqslant L\left(1 + |z|^{p-1}\right). \tag{3.9}$$

It is also possible to show (see [17]) that g also satisfies

$$D^2 g(z)\lambda \otimes \lambda \geqslant \nu_2\left(\mu^2 + |z|^2\right)^{(p-2)/2} |\lambda|^2, \tag{3.10}$$

with $\nu_2 \equiv \nu_2(\gamma_1, \gamma_2, L) > 0$ and $0 \leqslant \mu \leqslant 1$.

A reference estimate

Proposition 3.3. *Let $g:\mathbb{R}^n\to\mathbb{R}$ be a function of class C^2 satisfying* (H6) *and* (H7). *Moreover let v be a local minimizer in K of the functional* (3.6) *with $B_R\Subset\Omega$. If in addition the function ψ fulfills* (2.7) *for some $n-\gamma_1<\lambda<n$, then for all $0<\rho<R/2$ and any $\varepsilon>0$ there holds*

$$\int_{B_\rho}\left|Dv(x)\right|^p dx\leqslant c\left[\left(\frac{\rho}{R}\right)^n+\varepsilon\right]\int_{B_R}\left(1+\left|Dv(x)\right|^p\right)dx+\bar{c}R^\lambda,$$

where $c\equiv c(\gamma_1,\gamma_2,L)$, $\bar{c}\equiv\bar{c}(\gamma_1,\gamma_2,L,\varepsilon)$ and λ denotes the exponent appearing in (2.7).

Proof. The proof of this result can be carried out as in Proposition 4.1 of [15]. One indeed has to make sure that the constants involved only depend on the global bounds γ_1 and γ_2 of the exponent function p. In this respect, the key points are Theorem 2.2 of [17] and the structure conditions (3.8)–(3.10). □

An up-to-the-boundary higher integrability result

Proposition 3.4. *Let $g:\mathbb{R}^n\to\mathbb{R}$ be a continuous function fulfilling* (H6) *for some $\gamma_1\leqslant p\leqslant\gamma_2$. Let v be a local minimizer of the functional* (3.6) *in the Dirichlet class $\{v\in u+W_0^{1,p}(B_R): v\geqslant\psi\}$, for some $u\in W^{1,p}(B_R)$, where the function ψ fulfills the assumption* (2.7). *If moreover $u\in W^{1,\tilde{p}}(B_R)$ for a certain $p<\tilde{p}$, then there exist $\tilde{\varepsilon}\equiv\tilde{\varepsilon}(n,\gamma_1,\gamma_2,L)\in(0,\tilde{p}/p-1)$ and $c\equiv c(n,\gamma_1,\gamma_2,L)$ such that*

$$\left(\fint_{B_R}|Dv|^{p(1+\tilde{\varepsilon})}dx\right)^{\frac{1}{p(1+\tilde{\varepsilon})}}\leqslant c\left[\left(\fint_{B_R}|Dv|^p dx\right)^{1/p}+\left(\fint_{B_R}\left(|Du|^{\tilde{p}}+|D\psi|^{\tilde{p}}+1\right)dx\right)^{1/\tilde{p}}\right].\tag{3.11}$$

Proof. The proof is in many stages similar to the proof in [25]. Nevertheless, for convenience of the reader we will point out the main steps and especially the changes due to the obstacle.

Case 1 *(Interior situation).* Let $0<\rho<R$ and $x_0\in B_R$ be an interior point such that $B_\rho(x_0)\subset B_R$. Let $t,s\in\mathbb{R}$ with $\frac{\rho}{2}<t<s<\rho$. Let $\eta\in C_c^\infty(B_\rho)$, $0\leqslant\eta\leqslant1$ be a cut-off function with $\eta\equiv1$ on B_t, $\eta\equiv0$ outside B_s and $|D\eta|\leqslant\frac{2}{|s-t|}$. We define the functions $\tilde{v}:=v-\eta(v-(v)_\rho)$ and $w:=\max\{\tilde{v},\psi\}$. The function w is an admissible test function in (3.6).

We define the set $\Sigma:=\{x\in\mathbb{R}^n:\ \tilde{v}(x)\geqslant\psi(x)\}$. Testing the minimality with w and proceeding similarly as in the proof of Lemma 3.2, exploiting the growth conditions, we deduce

$$\begin{aligned}\int_{B_t}|Dv|^p dx&\leqslant L\int_{B_s}g(Dv)\,dx\leqslant L\int_{B_s}g(Dw)\,dx\\&=L\left[\int_{B_s\cap\Sigma}g(Dw)\,dx+\int_{B_s\setminus\Sigma}g(Dw)\,dx\right]\\&=L\left[\int_{B_s\cap\Sigma}g(D\tilde{v})\,dx+\int_{B_s\setminus\Sigma}g(D\psi)\,dx\right]\\&\leqslant L\left[\int_{B_s}g(D\tilde{v})\,dx+\int_{B_s}g(D\psi)\,dx\right]\\&\leqslant L^2\int_{B_s}\left(1+\left|D\left(v-\eta\left(v-(v)_\rho\right)\right)\right|^p\right)dx+L^2\int_{B_s}\left(1+|D\psi|^p\right)dx.\end{aligned}$$

Using the properties of the cut-off function we find

$$\int_{B_t}|Dv|^p dx\leqslant c\int_{B_s\setminus B_t}|Dv|^p dx+\frac{c}{|s-t|^p}\int_{B_\rho}\left|v-(v)_\rho\right|^p dx+c\int_{B_\rho}\left(1+|D\psi|^p\right)dx.$$

'Filling the hole,' applying [21, Lemma 6.1] and using Sobolev–Poincaré's inequality, we deduce the following reverse Hölder inequality:

$$\fint_{B_{\rho/2}} |Dv|^p\,dx \leqslant c_1\left(\fint_{B_\rho} |Dv|^{p\chi}\,dx\right)^{1/\chi} + c_2 \fint_{B_\rho}\left(1+|D\psi|^p\right)dx, \tag{3.12}$$

with $\chi := \frac{n}{n+p} < 1$, $c_1, c_2 \equiv c_1, c_2(n, \gamma_1, \gamma_2, L)$.

Case 2 *(Situation at the boundary).* We consider a point $x_0 \in \partial B_R$ and $0 < \rho < R$. Using the same cut-off function as before, we define $\tilde{v} := v - \eta(v-u)$ and $w := \max\{\tilde{v}, \psi\}$.

On ∂B_R we have $\tilde{v} \equiv u \geqslant \psi$ and therefore $w|_{\partial B_R} \equiv u$ which yields that $w \in u + W_0^{1,p}(B_R)$, $w \geqslant \psi$ is an admissible test function in (3.6). Additionally we have $w = \max\{v, \psi\}$ on ∂B_ρ and on $B_\rho \setminus B_s$, and $w = u$ on B_t. Defining $B_t^+ := B_t(x_0) \cap B_R$ and testing the minimality in (3.6) we obtain in exactly the same way as before

$$\begin{aligned}
\int_{B_t^+} |Dv|^p\,dx &\leqslant L \int_{B_s^+\cap\Sigma} g(Dw)\,dx + L \int_{B_s^+\setminus\Sigma} g(D\psi)\,dx \\
&\leqslant L^2 \int_{B_s^+}\left(1+|D\tilde{v}|^p\right)dx + L^2 \int_{B_\rho^+}\left(1+|D\psi|^p\right)dx \\
&\leqslant c\left[\int_{B_s^+\setminus B_t^+} |Dv|^p\,dx + \int_{B_s^+} |Du|^p\,dx + \frac{1}{(s-t)^p}\int_{B_s^+} |v-u|^p\,dx\right] + c\int_{B_\rho^+}\left(1+|D\psi|^p\right)dx.
\end{aligned}$$

Again ‘filling the hole’ and using [21, Lemma 6.1], we obtain

$$\int_{B_{\rho/2}^+} |Dv|^p\,dx \leqslant c\left[\int_{B_\rho^+}\left|\frac{v-u}{\rho}\right|^p dx + \int_{B_\rho^+} |Du|^p\,dx + \int_{B_\rho^+}\left(1+|D\psi|^p\right)dx\right].$$

Defining

$$\bar{w} := \begin{cases} v-u & \text{on } B_\rho^+, \\ 0 & \text{on } B_\rho^- := B_\rho \setminus B_\rho^+, \end{cases}$$

and applying Sobolev–Poincaré’s inequality in the version of [31, Corollary 4.5.3] (note that $|B_\rho^-| \geqslant 1/2|B_\rho|$) we deduce

$$\int_{B_\rho^+} |v-u|^p\,dx = \int_{B_\rho} |\bar{w}|^p\,dx \leqslant c(n,\gamma_2)\frac{|B_\rho|}{|B_\rho^-|}\left(\int_{B_\rho} |D\bar{w}|^{\frac{np}{n+p}}\,dx\right)^{\frac{n+p}{n}} \leqslant c(n,\gamma_2)\left(\int_{B_\rho^+} \left|D(v-u)\right|^{\frac{np}{n+p}}\,dx\right)^{\frac{n+p}{n}}.$$

Defining $\chi := \frac{n}{n+p} < 1$, taking mean values and using Hölder’s inequality we end up with

$$\fint_{B_{\rho/2}^+} |Dv|^p\,dx \leqslant c\left[\left(\fint_{B_\rho^+} |Dv|^{p\chi}\,dx\right)^{1/\chi} + \fint_{B_\rho^+}\left(|Du|^p + |D\psi|^p + 1\right)dx\right], \tag{3.13}$$

with $c \equiv c(n, \gamma_2, L)$. Note that (3.12) holds for any $B_\rho \subset B_R$ and (3.13) for any $0 < \rho \leqslant R$. Therefore we can apply a global version of Gehring’s Lemma [9, Theorem 2.4], with the functions $g := |Dv|^{p\chi}$, $f := (|Du|^p + |D\psi|^p + 1)^\chi$ to deduce the desired result. $\square$

Remark. Note that the dependency of the higher integrability exponent $\tilde{\varepsilon}$ and the constants coming up in Gehring’s Lemma on the exponent p can be replaced by dependencies on the global bounds γ_1 and γ_2 for p. For a detailed discussion of this we refer the reader to [24].

Iteration lemma

We will use the following iteration lemma, which is a slight modification of Lemma 7.3 in [21] and can be found in this version in [20, Chapter 3.2], for the proof of our results.

Lemma 3.5. *Let $\Phi(t)$ be a nonnegative and nondecreasing function. Suppose that*

$$\Phi(\rho) \leqslant A\left[\left(\frac{\rho}{R}\right)^{\alpha} + \varepsilon\right]\Phi(R) + BR^{\beta},$$

for all $\rho \leqslant R \leqslant R_0$, with A, B, α, β nonnegative constants, $\beta < \alpha$. Then there exists a constant $\varepsilon_0 \equiv \varepsilon_0(A, \alpha, \beta)$ such that if $\varepsilon < \varepsilon_0$, for all $\rho \leqslant R \leqslant R_0$, then

$$\Phi(\rho) \leqslant c\left[\left(\frac{\rho}{R}\right)^{\beta}\Phi(R) + B\rho^{\beta}\right],$$

where c is a constant depending on α, β, A, but independent of B.

Technical lemmas

At several stages in this paper we will make use of the following two well-known technical lemmas.

Lemma 3.6. *(See [6, Lemma 2.2].) If $p > 1$ is such that there exist two constants γ_1, γ_2 with $\gamma_1 \leqslant p \leqslant \gamma_2$, then there exists a constant $c \equiv c(\gamma_1, \gamma_2)$ such that for any $\mu \geqslant 0$, $\xi, \eta \in \mathbb{R}^n$,*

$$\left(\mu^2 + |\xi|^2\right)^{p/2} \leqslant c\left(\mu^2 + |\eta|^2\right)^{p/2} + c\left(\mu^2 + |\xi|^2 + |\eta|^2\right)^{(p-2)/2}|\xi - \eta|^2.$$

Lemma 3.7. *(See [21, Lemma 8.3].) Let $\xi, \eta \in \mathbb{R}^n$ and $Z(t) = (1 + |(1-t)\xi + t\eta|^2)^{1/2}$. For every $s > -1$ and $r > 0$ there exist two constants $c_1, c_2 \equiv c_1, c_2(s, r)$ such that*

$$c_1\left(1 + |\xi|^2 + |\eta|^2\right)^{s/2} \leqslant \int_0^1 (1-t)^r Z(t)^s\,dt \leqslant c_2\left(1 + |\xi|^2 + |\eta|^2\right)^{s/2}.$$

4. Proof of Theorem 2.8

A priori assumptions

For the proof of Theorem 2.8 we will assume that the modulus of continuity of the growth exponent p satisfies condition (2.8). Therefore, in particular we may always assume (2.2). Additionally, since our results are local in nature, we may assume that the obstacle ψ fulfills a global Morrey condition, i.e. there exists a constant $c < +\infty$ such that

$$\|D\psi\|_{L^{q,\lambda}(\Omega)} \leqslant c.$$

Step 1 *(Localization).* Let us start with Lemma 3.2 which provides a higher integrability exponent δ such that for any $\Omega' \Subset \Omega$ there holds

$$\int_{\Omega'} |Du|^{p(x)(1+\delta)}\,dx < +\infty.$$

Of course we can choose $\delta < r - 1$, where r is the quantity appearing in (2.7). Let us assume that the $p(x)$ energy on Ω is bounded, i.e. that there exists a constant M such that

$$\int_{\Omega} |Du|^{p(x)}\,dx \leqslant M < +\infty. \tag{4.1}$$

In the sequel we will explicitly point out if constants depend on this bound M.

Further localization. Let R_M be a maximal radius such that there holds $\omega_1(8R_M) \leqslant \delta/4$. Let $\mathcal{O} \Subset \Omega$ be a set whose diameter does not exceed R_M. We denote

$$p_2 := \max\left\{p(x)\colon\ x \in \overline{\mathcal{O}}\right\} = p(x_0), \qquad p_1 := \min\left\{p(x)\colon\ x \in \overline{\mathcal{O}}\right\}. \tag{4.2}$$

Then there holds

$$p_2 - p_1 \leqslant \omega_1(8R_M) \leqslant \delta/4; \tag{4.3}$$

$$p_2(1 + \delta/4) \leqslant p(x)(1 + \delta/4 + \omega_1(R)) \leqslant p(x)(1 + \delta). \tag{4.4}$$

Furthermore we note that the localization together with the bound (2.2) for the modulus of continuity provides for any $R \leqslant 8R_M \leqslant 1$:

$$R^{-n\omega_1(R)} \leqslant \exp(nL) = c(n, L), \qquad R^{-\frac{n\omega_1(R)}{1+\omega_1(R)}} \leqslant c(n, L). \tag{4.5}$$

Higher integrability. By our higher integrability result and the localization, we immediately obtain

$$\fint_{B_R} |Du|^{p_2}\,dx \leqslant c\left[\left(\fint_{B_{2R}} |Du|^{p(x)}\,dx\right)^{1+\delta/4} + \fint_{B_{2R}} |D\psi|^{p(x)(1+\delta/4)}\,dx + 1\right]. \tag{4.6}$$

Step 2 *(Freezing).* Let B_R be a ball in $\mathcal{O}$. We define $v \in u + W_0^{1,p_2}(B_R)$ as the unique minimizer of the functional

$$\mathcal{G}(v) := \int_{B_R} h(x_0, Dv)\,dx =: \int_{B_R} g(Dv)\,dx$$

in the class $\{v \in u + W_0^{1,p_2}(B_R)\colon\ v \geqslant \psi\}$. Since the functional $\mathcal{G}$ is frozen in the point x_0, it satisfies the growth and ellipticity conditions (H6) and (H7) with the maximal exponent $p = p_2$. For our proof we will assume that $g \in \mathcal{C}^2$. Removing the $\mathcal{C}^2$ regularity of g can then be done by a suitable approximation, arguing exactly as in [2]. Note that by the minimizing property of v and the growth conditions (H6), we obtain the following estimate for the p_2 energy of v (since $u \in K$):

$$\int_{B_R} |Dv|^{p_2}\,dx \leqslant L^2 \int_{B_R} \left(1 + |Du|^{p_2}\right) dx < +\infty. \tag{4.7}$$

Reference estimate. v is a K-minimizer of the frozen functional with constant p_2 growth. Therefore it satisfies the assumptions of Proposition 3.3 with $1 < \gamma_1 \leqslant p \equiv p_2 \leqslant \gamma_2$. Thus there holds for any $\varepsilon > 0$ and any ρ with $2\rho < R$,

$$\int_{B_\rho} |Dv|^{p_2}\,dx \leqslant c\left[\left(\frac{\rho}{R}\right)^n + \varepsilon\right] \int_{B_R} \left(1 + |Dv|^{p_2}\right) dx + \bar{c}R^\lambda, \tag{4.8}$$

with $c \equiv c(\gamma_1, \gamma_2, L)$ and $\bar{c} \equiv \bar{c}(\gamma_1, \gamma_2, L, \varepsilon)$.

Comparison estimate. We prove the following comparison estimate:

$$\int_{B_R} \left(\mu^2 + |Du|^2 + |Dv|^2\right)^{\frac{p_2-2}{2}} |Du - Dv|^2\,dx \leqslant c\omega_1(R)\log\left(\frac{1}{R}\right)\left[\int_{B_{2R}} \left(1 + |Du|^{p_2}\right) dx + R^\lambda\right], \tag{4.9}$$

with a constant $c \equiv c(n, \gamma_1, \gamma_2, L, M, \delta, r)$, where r is the exponent appearing in (2.7). Using the differentiability of g and the ellipticity (3.10) with $p = p_2$, we estimate

$$\begin{aligned}
\mathcal{G}(u) - \mathcal{G}(v) &= \int_{B_R} \left[g(Du) - g(Dv)\right] dx \\
&= \int_{B_R} \left\langle Dg(Dv), Du - Dv\right\rangle dx \quad [=0] \\
&\quad + \int_{B_R} \int_0^1 \left\langle (1-t)D^2 g\big(tDu + (1-t)Dv\big)(Du - Dv), (Du - Dv)\right\rangle dt\,dx \\
&\geqslant \nu_2 \int_{B_R} \int_0^1 (1-t)\left(\mu^2 + \left|tDu + (1-t)Dv\right|^2\right)^{(p_2-2)/2} |Du - Dv|^2\,dt\,dx \\
&\geqslant c^{-1} \int_{B_R} \left(\mu^2 + |Du|^2 + |Dv|^2\right)^{(p_2-2)/2} |Du - Dv|^2\,dx,
\end{aligned} \tag{4.10}$$

with $c \equiv c(\gamma_1, \gamma_2, L)$, where in the last inequality we used Lemma 3.7. On the other hand we have

$$\begin{aligned}
\int_{B_R} \left[g(Du) - g(Dv)\right] dx &= \int_{B_R} \left[h(x_0, Du) - h(x, Du)\right] dx + \int_{B_R} \left[h(x, Du) - h(x, Dv)\right] dx + \int_{B_R} \left[h(x, Dv) - h(x_0, Dv)\right] dx \\
&= I^{(1)} + I^{(2)} + I^{(3)}.
\end{aligned}$$

Obviously we have $I^{(2)} \leqslant 0$, since u is a minimizer of the functional (2.6) in the class K and $v \geqslant \psi$. We estimate $I^{(1)}$, using the continuity of the integrand with respect to the variable x,

$$I^{(1)} \leqslant c \int_{B_R} \omega_1\big(|x-x_0|\big)\big(\big(\mu^2+|Du|^2\big)^{p(x)/2} + \big(\mu^2+|Du|^2\big)^{p_2/2}\big)\big(1+\big|\log\big(\mu^2+|Du|^2\big)\big|\big)\,dx.$$

Decomposing B_R into the sets $B_R^+ := B_R \cap \{|Du| \geqslant e\}$ and $B_R^- := B_R \cap \{|Du| < e\}$ we argue as follows: on the set B_R^- we have that $(\mu^2+|Du|^2)^{p_2/2}(1+|\log(\mu^2+|Du|^2)|) \leqslant c(\gamma_1,\gamma_2)$, whereas on the set B_R^+, making use of the fact that $\mu \leqslant e \leqslant |Du|$ and of the elementary inequality $\log(e+ab) \leqslant \log(e+a)+\log(e+b)$, for all $a,b>0$, which is a direct consequence of the concavity of the logarithm, we estimate as follows:

$$\begin{aligned}\int_{B_R^+} \big(\mu^2+|Du|^2\big)^{p_2/2}\big|\log\big(\mu^2+|Du|^2\big)\big|\,dx &\leqslant c(\gamma_1,\gamma_2)\int_{B_R^+} |Du|^{p_2}\log\big(e+|Du|^{p_2}\big)\,dx \\ &\leqslant cR^n \fint_{B_R} |Du|^{p_2}\log\big(e+\big\||Du|^{p_2}\big\|_{L^1(B_R)}\big)\,dx + c\int_{B_R} |Du|^{p_2}\log\bigg(e+\frac{|Du|^{p_2}}{\||Du|^{p_2}\|_{L^1(B_R)}}\bigg)\,dx\end{aligned}$$

which gives us the splitting

$$\begin{aligned}I^{(1)} &\leqslant c\omega_1(R)R^n \fint_{B_R} |Du|^{p_2}\log\big(e+\big\||Du|^{p_2}\big\|_{L^1(B_R)}\big)\,dx + c\omega_1(R)\int_{B_R} |Du|^{p_2}\log\bigg(e+\frac{|Du|^{p_2}}{\||Du|^{p_2}\|_{L^1(B_R)}}\bigg)\,dx + c\omega_1(R)R^n \\ &= I_1^{(1)}+I_2^{(1)}+I_3^{(1)}. \end{aligned} \tag{4.11}$$

We estimate $I_2^{(1)}$, using first (3.3) in [2], which is a basic estimate for the $L\log L$ norm, then exploiting higher integrability (3.2) with exponent δ which we had chosen less than $r-1$ (see also the remark after Lemma 3.2), the localization (4.4) and the bound M for the $p(x)$ energy (4.1):

$$\begin{aligned}I_2^{(1)} &\leqslant c(p_2,\delta)\omega_1(R)R^n\bigg(\fint_{B_R} |Du|^{p_2(1+\delta/4)}\,dx\bigg)^{1/(1+\delta/4)} \\ &\leqslant c\omega_1(R)R^n + c\omega_1(R)R^n\bigg(\fint_{B_R} |Du|^{p(x)(1+\delta/4+\omega_1(R))}\,dx\bigg)^{1/(1+\delta/4)} \\ &\leqslant c\omega_1(R)R^n + c(\delta)\omega_1(R)R^n\bigg[\bigg(\fint_{B_{2R}} |Du|^{p(x)}\,dx\bigg)^{\frac{1+\delta/4+\omega_1(R)}{1+\delta/4}} + \bigg(\fint_{B_{2R}} \big(|D\psi|^{p(x)(1+\delta)}+1\big)\,dx\bigg)^{1/(1+\delta/4)}\bigg] \\ &\overset{(2.7)}{\leqslant} c\omega_1(R)R^n + c\omega_1(R)R^nR^{-n\frac{\omega_1(R)}{1+\delta/4}}\bigg(\fint_{B_{2R}} |Du|^{p(x)}\,dx\bigg)\bigg(\int_{B_{2R}} |Du|^{p(x)}\,dx\bigg)^{\frac{\omega_1(R)}{1+\delta/4}} + c\omega_1(R)R^nR^{\frac{(\lambda-n)}{1+\delta/4}} \\ &\overset{(4.5)}{\leqslant} c\omega_1(R)R^n + c\omega_1(R)R^n\bigg(\fint_{B_{2R}} \big(1+|Du|^{p_2}\big)\,dx\bigg)\bigg(\int_{B_{2R}} |Du|^{p(x)}\,dx\bigg)^{\frac{\omega_1(R)}{1+\delta/4}} + c\omega_1(R)R^\lambda R^{(n-\lambda)\frac{\delta/4}{1+\delta/4}} \\ &\leqslant c\omega_1(R)\cdot M\cdot\int_{B_{2R}} \big(1+|Du|^{p_2}\big)\,dx + c\omega_1(R)R^\lambda,\end{aligned}$$

with $c \equiv c(n,L,\gamma_1,\gamma_2,r,\delta)$. In the last step we used the bound M for the $p(x)$ energy of u and the facts that $R \leqslant 1$ and $\lambda < n$. We estimate $I_1^{(1)}$, using elementary estimates for the logarithm and the fact that for any $\delta>0$ we have $\log(e+z) \leqslant c(\delta)(1+z)^\delta$:

$$\begin{aligned}I_1^{(1)} &\leqslant c\omega_1(R)\log\bigg(R^{-n}e + R^{-n}\int_{B_R} |Du|^{p_2}\,dx\bigg)\int_{B_R} |Du|^{p_2}\,dx \\ &\leqslant c\omega_1(R)\int_{B_R} |Du|^{p_2}\,dx\cdot\log\bigg(e+\int_{B_R} |Du|^{p_2}\,dx\bigg) + c\omega_1(R)\log\bigg(\frac{1}{R}\bigg)\int_{B_R} |Du|^{p_2}\,dx \\ &\leqslant c(\delta)\omega_1(R)\bigg(1+\int_{B_R} |Du|^{p_2}\,dx\bigg)^{\delta/4}\int_{B_R} |Du|^{p_2}\,dx + c\omega_1(R)\log\bigg(\frac{1}{R}\bigg)\int_{B_R} |Du|^{p_2}\,dx\end{aligned}$$

$$\leqslant c(M,n,\delta)\omega_1(R)\log\left(\frac{1}{R}\right)\int\limits_{B_R}\left(1+|Du|^{p_2}\right)dx.$$

Thus, alltogether we obtain

$$I^{(1)} \leqslant c\omega_1(R)\log\left(\frac{1}{R}\right)\left[\int\limits_{B_{2R}}\left(1+|Du|^{p_2}\right)dx + R^{\lambda}\right],$$

with $c \equiv c(n,\gamma_1,\gamma_2,L,M,\delta,r)$. We handle $I^{(3)}$ in a similar way to $I^{(1)}$. Estimating in exactly the same way as in (4.11) with v instead of u and doing the same splitting into $I_1^{(3)}$ to $I_3^{(3)}$, we use higher integrability up to the boundary for v (3.11) (with $p \equiv p_2$, $\tilde{p} \equiv p_2(1+\delta/4)$ and $\tilde{\varepsilon} \in (0,\delta/4)$) and the estimate (4.7) for the p_2 energy of v:

$$\begin{aligned}
I_2^{(3)} &\leqslant c\omega_1(R)R^n\left(\fint\limits_{B_R}|Dv|^{p_2(1+\tilde{\varepsilon})}\,dx\right)^{1/(1+\tilde{\varepsilon})}\\
&\leqslant c\omega_1(R)R^n\left[\fint\limits_{B_R}|Dv|^{p_2}\,dx + \left(\fint\limits_{B_R}\left(|Du|^{p_2(1+\delta/4)}+1\right)dx\right)^{\frac{1}{1+\delta/4}} + \left(\fint\limits_{B_R}\left(|D\psi|^{p_2(1+\delta/4)}+1\right)dx\right)^{\frac{1}{1+\delta/4}}\right]\\
&\leqslant c\omega_1(R)\int\limits_{B_R}\left(1+|Du|^{p_2}\right)dx + c\omega_1(R)R^n\left(\fint\limits_{B_R}|Du|^{p_2(1+\delta/4)}\,dx\right)^{\frac{1}{1+\delta/4}} + c\omega_1(R)R^{\lambda}R^{(n-\lambda)\frac{\delta/4}{1+\delta/4}}\\
&\leqslant c(M)\omega_1(R)\int\limits_{B_{2R}}\left(1+|Du|^{p_2}\right)dx + c\omega_1(R)R^{\lambda}.
\end{aligned}$$

In the last step we used the estimate for $I_2^{(1)}$ to handle the second term and again the facts that $R \leqslant 1$ and $\lambda < n$. $I_1^{(3)}$ is estimated in exactly the same way as $I_1^{(1)}$, additionally using (4.7) for passing over from the p_2 energy of v to the p_2 energy of u. Alltogether we end up with

$$I^{(3)} \leqslant c\left(\omega_1(R)\log\left(\frac{1}{R}\right)\right)\left[\int\limits_{B_{2R}}\left(1+|Du|^{p_2}\right)dx + R^{\lambda}\right],$$

with $c \equiv c(n,\gamma_1,\gamma_2,L,M,\delta,r)$. Taking the estimates for $I^{(1)}$ to $I^{(3)}$ together, we end up with the desired comparison estimate (4.9).

Conclusion. Now we put together our reference estimate and the comparison estimate to deduce a decay estimate for the p_2 energy of u. By the Technical Lemma 3.6 we now split as follows:

$$\begin{aligned}
\int\limits_{B_\rho}|Du|^{p_2}\,dx &\leqslant \int\limits_{B_\rho}\left(\mu^2+|Du|^2\right)^{p_2/2}dx\\
&\leqslant c\int\limits_{B_\rho}\left(\mu^2+|Dv|^2\right)^{p_2/2}dx + c\int\limits_{B_\rho}\left(\mu^2+|Du^2|+|Dv|^2\right)^{\frac{p_2-2}{2}}|Du-Dv|^2\,dx\\
&=: \mathcal{A}+\mathcal{B}.
\end{aligned}$$

For $\mathcal{A}$ we use the reference estimate (4.8) and estimate (4.7) to deduce (note that $\rho \leqslant 1$)

$$\begin{aligned}
\mathcal{A} &\leqslant c\rho^n + \int\limits_{B_\rho}|Dv|^{p_2}\,dx\\
&\leqslant c\left[\left(\frac{\rho}{R}\right)^n+\varepsilon\right]\int\limits_{B_R}\left(1+|Dv|^{p_2}\right)dx + cR^{\lambda}\\
&\leqslant c\left[\left(\frac{\rho}{R}\right)^n+\varepsilon\right]\int\limits_{B_R}\left(1+|Du|^{p_2}\right)dx + cR^{\lambda}.
\end{aligned}$$

For the term $\mathcal{B}$ we use the comparison estimate (4.9)

$$\mathcal{B} \leqslant c\omega_1(R)\log\left(\frac{1}{R}\right)\left[\int\limits_{B_{2R}}\left(1+|Du|^{p_2}\right)dx + R^{\lambda}\right]. \tag{4.12}$$

Thus, alltogether we end up with

$$\int_{B_\rho} |Du|^{p_2}\,dx \leqslant c\left[\left(\frac{\rho}{R}\right)^n + \varepsilon + \omega_1(R)\log\left(\frac{1}{R}\right)\right]\int_{B_{2R}} \left(1+|Du|^{p_2}\right)dx + \bar{c}R^\lambda, \tag{4.13}$$

where $c \equiv c(n, L, M, \gamma_1, \gamma_2, r)$ and $\bar{c} \equiv \bar{c}(n, M, L, \gamma_1, \gamma_2, \varepsilon, r)$.

Step 3 *(Proof of the theorem).* Let B_{R_0} be a ball whose radius is small enough to satisfy $R_0 \leqslant R_M$. Then estimate (4.13) holds for all radii $0 < \rho \leqslant R \leqslant R_0$. Let $\varepsilon_0 \equiv \varepsilon_0(n, M, L, \gamma_1, \gamma_2, \lambda) > 0$ be the quantity provided by Lemma 3.5. We choose $\varepsilon \equiv \varepsilon_0/2$. This fixes the dependencies of the constant in (4.13), i.e. $\bar{c} \equiv \bar{c}(n, L, M, \gamma_1, \gamma_2, \lambda, r)$. Then by assumption (2.8) we find a radius $R_1 > 0$ so small that $\omega_1(R_1)\log(1/R_1) < \varepsilon_0/2$, thus

$$\omega_1(R)\log\left(\frac{1}{R}\right) + \varepsilon < \varepsilon_0,$$

for any $0 < R \leqslant R_1$ and therefore we have $R_1 \equiv R_1(n, \gamma_1, \gamma_2, L, M, \omega_1, \lambda)$. Lemma 3.5 then yields

$$\int_{B_\rho} \left|Du(x)\right|^{p_2}dx \leqslant c\rho^\lambda,$$

with $c \equiv c(n, M, L, \gamma_1, \gamma_2, \lambda, r)$, whenever $0 < \rho < R_1$. Since we have $\gamma_1 \leqslant p_2 \leqslant \gamma_2$, we deduce by a standard covering argument that

$$Du \in L^{\gamma_1,\lambda}_{\mathrm{loc}}(\Omega),$$

and thus by Poincaré's inequality we conclude that $u \in C^{0,\alpha}_{\mathrm{loc}}(\Omega)$ with $\alpha = 1 - \frac{n-\lambda}{\gamma_1}$.

5. Proof of Theorem 2.9

First we remark that, since we are proving only local results, by Lemma 3.1 we may assume that our minimizer u of (1.1) is globally Hölder continuous, i.e. there exists $\gamma \in (0, 1)$ such that

$$\left|u(x) - u(y)\right| \leqslant [u]_\gamma |x-y|^\gamma \leqslant c|x-y|^\gamma, \tag{5.1}$$

for all $x, y \in \Omega$.

As in the proof of Theorem 2.8, we may assume that the obstacle ψ fulfills a global Morrey condition, i.e. that there exists a constant $c < +\infty$ such that

$$\|D\psi\|_{L^{q,\lambda}(\Omega)} \leqslant c. \tag{5.2}$$

We start with a technical lemma which we will need later in the proof. The proof of a slightly modified lemma can be found in [15].

Proposition 5.1. *Let $g : \mathbb{R}^n \to \mathbb{R}$ be a continuous function satisfying* (H6) *and* (H7) *with $1 < \gamma_1 \leqslant p \leqslant \gamma_2 < \infty$. Let $u \in K$, $B_R \Subset \Omega$ and let us assume that there exists $v_0 \in u + W^{1,p}_0(B_R)$ being a minimizer of the functional*

$$\mathcal{H}(w, B_R) := \int_{B_R} g\big(Dw(x)\big)\,dx + \theta_0\left(\int_{B_R} |Dw - Dv_0|^p\,dx\right)^{1/p}$$

in the Dirichlet class

$$\overline{D} := \left\{w \in u + W^{1,p}_0(B_R),\ w \geqslant \psi\right\},$$

where $\theta_0 \geqslant 0$. Then, for all $\beta > 0$, for all $A_0 > 0$ and for any $\varepsilon > 0$ we have

$$\begin{aligned}\int_{B_\rho} \left|Dv_0(x)\right|^p dx &\leqslant c\left[\left(\frac{\rho}{R}\right)^n + \varepsilon\right]\int_{B_R}\left(1+\left|Dv_0(x)\right|^p\right)dx + \bar{c}R^\lambda + c\theta_0\left(\int_{B_R}\left|Du(x) - Dv_0(x)\right|^p dx\right)^{1/p} + c\theta_0^{\frac{p}{p-1}}\left[\frac{1}{A_0}\right]^{\frac{p\beta}{p-1}}\\ &\quad + c[A_0]^{p\beta}\int_{B_R}\left(1+\left|Du(x)\right|^p\right)dx,\end{aligned}$$

for any $0 < \rho < R/2$, where the constants c depend only on L, γ_1, γ_2 while the constant $\bar{c}$ depends also on ε.

Step 1 *(Localizing).* As in the proof of Theorem 2.8, we start with Lemma 3.2 which provides a higher integrability exponent δ_1 such that for any $\Omega' \Subset \Omega$ there holds

$$\int_{\Omega'} |Du|^{p(x)(1+\delta_1)}\,dx < +\infty. \tag{5.3}$$

Again we assume that there exists a constant M such that

$$\int_{\Omega} |Du|^{p(x)}\,dx \leqslant M < +\infty. \tag{5.4}$$

The up-to-the-boundary higher integrability result Proposition 3.4 provides a second exponent $\delta_2 > 0$. For the rest of the proof we define

$$\delta := \min\left\{\delta_1, \delta_2, r-1, \frac{\gamma}{1-\gamma}, \frac{p_2+\lambda-n}{n-\lambda}\right\},$$

where r is the quantity appearing in assumption (2.7). We moreover set

$$\tilde m := \min\left\{\frac{\lambda-n}{p_2}+1, 1-\frac{n-\lambda}{p_2}(1+\delta), \gamma+\gamma\delta-\delta\right\} \tag{5.5}$$

and due to our assumptions it turns out that $0<\tilde m<1$. Now let R_M be a radius such that $\omega_1(8R_M)+\omega_2(8R_M)\leqslant \delta/4$. Let $\mathcal{O}\Subset\Omega$ be a set whose diameter does not exceed R_M. As in the proof of Theorem 2.8 we define

$$p_2 := \max\{p(x)\colon\ x\in\overline{\mathcal{O}}\} = p(x_0), \qquad p_1 := \min\{p(x)\colon\ x\in\overline{\mathcal{O}}\}. \tag{5.6}$$

Then, again (4.4) holds, and therefore by higher integrability we deduce that

$$\int_{B_R} |Du|^{p_2}\,dx < +\infty. \tag{5.7}$$

Step 2 *(Freezing).* Let B_R be a ball in $\mathcal{O}$. We define $v\in u+W_0^{1,p_2}(B_R)$ as the unique solution of the minimizing problem

$$\min\left\{v\mapsto \mathcal{G}_0(v,B_R) := \int_{B_R} f\big(x_0,(u)_R,Dv(x)\big)\,dx,\ v\in u+W_0^{1,p_2}(B_R),\ v\geqslant\psi\right\}. \tag{5.8}$$

Since the functional $\mathcal{G}_0$ is frozen in the point $(x_0,(u)_R)$, it satisfies the growth and ellipticity conditions (H6) and (H7) with maximal exponent $p=p_2$. We note that since v is a minimizer of the functional $\mathcal{G}_0$ with boundary data u in ∂B_R, where $u|_{\partial B_R}$ is the trace of a Hölder continuous function, by Theorem 7.8 in [21] we conclude that $v\in\mathcal{C}^{0,\tilde\gamma}(\overline{B_R})$ for some $\tilde\gamma\in(0,1)$. Therefore, for the rest of the proof we assume that there exists $\gamma\in(0,1)$ such that

$$\big|v(x)-v(y)\big| \leqslant [v]_\gamma|x-y|^\gamma \leqslant c|x-y|^\gamma, \tag{5.9}$$

for all $x,y\in\overline{B_R}$. We remark that for simplicity we use the same Hölder exponent for the functions v and u (see (5.1)), which is not restrictive. Let us remark that, since v minimizes the functional (5.8), by the growth condition (H6), higher integrability and (5.7) we obtain the following estimate for the p_2 energy of v:

$$\int_{B_R} |Dv|^{p_2}\,dx \leqslant L^2\int_{B_R}\big(1+|Du|^{p_2}\big)\,dx < +\infty. \tag{5.10}$$

Step 3 *(Comparison estimate).* We will show that

$$\mathcal{G}_0(u)-\mathcal{G}_0(v) \leqslant c\left(\omega_1(R)\log\left(\frac{1}{R}\right)+\omega_2\big(R^{\tilde m}\big)\right)\left[\int_{B_{2R}}\big(1+|Du|^{p_2}\big)\,dx+R^\lambda\right], \tag{5.11}$$

with a constant $c\equiv c(n,\gamma_1,\gamma_2,M,L,\gamma,\lambda,r,[u]_\gamma,[v]_\gamma)$. Note here that M is the bound on the $p(x)$ energy which has been introduced in (5.4). Since u is a local minimizer in K of the functional (1.1), we obtain

$$\mathcal{G}_0(u,B_R) \leqslant \mathcal{G}_0(v,B_R) + \int_{B_R}\big[f\big(x_0,(u)_R,Du(x)\big)-f\big(x,u(x),Du(x)\big)\big]\,dx + \int_{B_R}\big[f\big(x,v(x),Dv(x)\big)-f\big(x_0,(u)_R,Dv(x)\big)\big]\,dx$$

$$
\begin{aligned}
&\leqslant \mathcal{G}_0(v, B_R) + \int_{B_R} \big[f\big(x_0, (u)_R, Du(x)\big) - f\big(x, (u)_R, Du(x)\big)\big]\,dx + \int_{B_R} \big[f\big(x, (u)_R, Du(x)\big) - f\big(x, u(x), Du(x)\big)\big]\,dx \\
&\quad + \int_{B_R} \big[f\big(x, v(x), Dv(x)\big) - f\big(x, (v)_R, Dv(x)\big)\big]\,dx + \int_{B_R} \big[f\big(x, (v)_R, Dv(x)\big) - f\big(x, (u)_R, Dv(x)\big)\big]\,dx \\
&\quad + \int_{B_R} \big[f\big(x, (u)_R, Dv(x)\big) - f\big(x_0, (u)_R, Dv(x)\big)\big]\,dx \\
&\leqslant \mathcal{G}_0(v, B_R) + I^{(4)} + I^{(5)} + I^{(6)} + I^{(7)} + I^{(8)},
\end{aligned}
$$

with the obvious labelling. At this point, the term $I^{(4)}$ can be estimated exactly as $I^{(1)}$, using first the continuity with respect to the first variable (H3) then splitting as in (4.11) and finally using estimates for the $L\log L$ norm, giving

$$
I^{(4)} \leqslant c\left(\omega_1(R)\log\left(\frac{1}{R}\right)\right)\left[\int_{B_{2R}} \big(1 + |Du|^{p_2}\big)\,dx + R^{\lambda}\right],
$$

with $c \equiv c(n, \gamma_1, \gamma_2, L, M, \delta, r)$. $I^{(8)}$ can be estimated exactly as the term $I^{(3)}$, giving

$$
I^{(8)} \leqslant c\left(\omega_1(R)\log\left(\frac{1}{R}\right)\right)\left[\int_{B_{2R}} \big(1 + |Du|^{p_2}\big)\,dx + R^{\lambda}\right],
$$

with $c \equiv c(n, \gamma_1, \gamma_2, L, \delta, r)$. Exploiting the continuity with respect to the second variable (H5) and Hölder continuity of u (5.1), we estimate $I^{(5)}$ as follows:

$$
I^{(5)} \leqslant L\int_{B_R} \omega\big(\big|u - (u)_R\big|\big)\big(\mu^2 + \big|Du(x)\big|^2\big)^{p(x)/2}\,dx \leqslant c\omega\big(R^{\gamma}\big)\int_{B_R} \big(1 + \big|Du(x)\big|^2\big)^{p_2/2}\,dx,
$$

with $c \equiv c(L, [u]_{\gamma})$. In a similar way we estimate $I^{(6)}$, first using the continuity with respect to the second variable (H5), then exploiting (5.9) and finally (5.10) as follows:

$$
I^{(6)} \leqslant L\int_{B_R} \omega_2\big(\big|v - (v)_R\big|\big)\big(\mu^2 + \big|Dv(x)\big|^2\big)^{p(x)/2}\,dx \leqslant c\omega_2\big(R^{\gamma}\big)\int_{B_R} \big(1 + \big|Du(x)\big|^{p_2}\big)\,dx,
$$

with a constant $c \equiv c(L, [v]_{\gamma})$. To treat the remaining term $I^{(7)}$, we first remark that

$$
\big|(u)_R - (v)_R\big| \leqslant \fint_{B_R} \big|u(x) - v(x)\big|\,dx. \tag{5.12}
$$

Therefore the main step consists in estimating the last term of the preceding inequality (5.12). First, using Poincaré's and Hölder's inequality, then (5.10) and finally higher integrability (4.6) we obtain

$$
\begin{aligned}
\fint_{B_R} \big|u(x) - v(x)\big|\,dx &\leqslant cR\fint_{B_R} \big|Du(x) - Dv(x)\big|\,dx \\
&\leqslant c\left(R^{p_2}\fint_{B_R} \big|Du(x) - Dv(x)\big|^{p_2}\,dx\right)^{1/p_2} \\
&\leqslant c\left(R^{p_2}\fint_{B_R} \big(1 + |Du|^{p_2}\big)\,dx\right)^{1/p_2} \\
&\overset{(4.6)}{\leqslant} cR + cR\left(\fint_{B_{2R}} |Du|^{p(x)}\,dx\right)^{\frac{1+\delta/4}{p_2}} + cR\left(\fint_{B_{2R}} |D\psi|^{p(x)(1+\delta/4)}\,dx\right)^{\frac{1}{p_2}},
\end{aligned}
$$

with $c \equiv c(n, r, \gamma_1, \gamma_2, L)$. The last term, containing the obstacle function ψ can be handled by (2.7), respectively (5.2) (note that $\delta \leqslant r - 1$) to conclude that

$$
\fint_{B_{2R}} |D\psi|^{p(x)(1+\delta/4)}\,dx \leqslant cR^{\lambda-n}.
$$

To treat the second term we use the Caccioppoli inequality (3.4) and Hölder continuity of u in terms of (5.1) together with the assumption (2.7) on the obstacle ψ to conclude

$$\begin{aligned}\fint_{B_{2R}} |Du|^{p(x)}\,dx &\leqslant c \fint_{B_{4R}} \left|\frac{u-(u)_{4R}}{R}\right|^{p(x)} dx + \fint_{B_{4R}} \big(1+|D\psi|^{p(x)}\big)\,dx \\ &\leqslant c\left[\fint_{B_{4R}} \left|\frac{u-(u)_{4R}}{R}\right|^{p_2} dx + R^{\lambda-n}+1\right] \\ &\leqslant c\big[[u]_\gamma^{p_2} R^{p_2(\gamma-1)} + R^{\lambda-n}+1\big].\end{aligned}$$

Taking these estimates together we deduce

$$\fint_{B_R} \big|u(x)-v(x)\big|\,dx \leqslant c\big[R + R^{(\gamma-1)(1+\delta)+1} + R^{\frac{(\lambda-n)(1+\delta)}{p_2}+1} + R^{\frac{\lambda-n}{p_2}+1}\big] \overset{(5.5)}{\leqslant} cR^{\tilde m},$$

with a constant $c \equiv c(n,\gamma_1,\gamma_2,L,[u]_\gamma,[v]_\gamma)$ and thus by (5.12) together with the monotonicity of ω_2

$$\omega_2\big(\big|(u)_R-(v)_R\big|\big) \leqslant c\omega_2\big(R^{\tilde m}\big). \tag{5.13}$$

Taking also into account (5.10), we have at this point

$$I^{(7)} \leqslant L\int_{B_R} \omega_2\big(\big|(u)_R-(v)_R\big|\big)\big(\mu^2+\big|Dv(x)\big|^2\big)^{p(x)/2}\,dx \leqslant c\omega_2\big(R^{\tilde m}\big)\int_{B_R}\big(1+|Du|^{p_2}\big)\,dx.$$

Collecting the previous bounds, summing up and taking into account that, since $\gamma \geqslant \gamma+\gamma\delta-\delta$ there holds $R^\gamma \leqslant R^{\tilde m}$, we obtain

$$I^{(4)}+\cdots+I^{(8)} \leqslant c\left(\omega_1(R)\log\left(\frac{1}{R}\right)+\omega_2\big(R^{\tilde m}\big)\right)\left[\int_{B_{2R}}\big(1+|Du|^{p_2}\big)\,dx + R^\lambda\right],$$

with $c \equiv c(n,\gamma_1,\gamma_2,M,L,\gamma,\lambda,r,[u]_\gamma,[v]_\gamma)$, which provides the desired estimate (5.11).

Step 4 *(Reference estimate and conclusion).* We set for simplicity

$$F(R) := \omega_1(R)\log\left(\frac{1}{R}\right)+\omega_2\big(R^{\tilde m}\big).$$

The assumption (2.9) allows us to say that

$$\lim_{R\to 0} F(R) = 0.$$

Now, by the minimality of v, we obtain

$$\mathcal{G}_0(u) \leqslant \inf_V \mathcal{G}_0 + H(R),$$

where we set

$$H(R) := cF(R)\left[\int_{B_{2R}}\big(1+\big|Du(x)\big|^{p_2}\big)\,dx + R^\lambda\right],$$

and

$$V = \big\{v \in u + W_0^{1,1}(B_R):\ v \geqslant \psi\big\}.$$

Let V be equipped with the distance

$$d(w_1,w_2) := H(R)^{-\frac{1}{p_2}}\left(\int_{B_R}\big|Dw_1(x)-Dw_2(x)\big|^{p_2}\,dx\right)^{1/p_2}. \tag{5.14}$$

Then (V,d) is a complete metric space. It is easy to see that the functional $\mathcal{G}_0$ is lower semicontinuous with respect to the topology induced by the distance d. Then by [13, Theorem 1] ("Ekeland variational principle") there exists $v_0 \in V$ such that

(i) $$\int_{B_R} \big|Du(x) - Dv_0(x)\big|^{p_2}\,dx \leqslant H(R),$$

(ii) $$\mathcal{G}_0(v_0) \leqslant \mathcal{G}_0(u),$$

(iii) v_0 is a local minimizer in V of the functional

$$w \mapsto \mathcal{H}(w) := \mathcal{G}_0(w) + \big[H(R)\big]^{\frac{p_2-1}{p_2}} \bigg(\int_{B_R} |Dw - Dv_0|^{p_2}\,dx\bigg)^{1/p_2}. \tag{5.15}$$

Remark 5.2. We choose to apply the Ekeland variational principle with the distance (5.14) which derives from a suitable weighted L^{p_2}-norm instead of the corresponding L^1-norm; the same trick has been successfully applied in the paper [27]. The advantage of this choice is that we can directly estimate the term

$$\int_{B_R} \big|Du(x) - Dv_0(x)\big|^{p_2}\,dx$$

by means of (5.15)(i) given by the Ekeland lemma without any further interpolation argument needed (which has been instead employed for example in [6] or [15]).

From the growth assumption (H6) with exponent $p = p_2$ and from property (5.15)(ii), as $u \in K$, we have

$$L^{-1} \int_{B_R} \big|Dv_0(x)\big|^{p_2}\,dx \leqslant \mathcal{G}_0(v_0) \leqslant \mathcal{G}_0(u) \leqslant L \int_{B_R} \big(1 + \big|Du(x)\big|^{p_2}\big)\,dx. \tag{5.16}$$

Now, we apply Proposition 5.1 with the following choices: $h(z) := f(x_0, (u)_R, z)$, $p = p_2$, $A_0 = F(R)$ and $\vartheta_0 = [H(R)]^{\frac{p_2-1}{p_2}}$ Then, by property (5.15)(i) and using (5.16), we have for every $\beta > 0$,

$$\begin{aligned}
\int_{B_\rho} \big|Dv_0(x)\big|^{p_2}\,dx &\leqslant c\bigg[\bigg(\frac{\rho}{R}\bigg)^n + \varepsilon\bigg] \int_{B_R} \big(1 + \big|Dv_0(x)\big|^{p_2}\big)\,dx + \bar{c}R^\lambda + c\big[H(R)\big]^{\frac{p_2-1}{p_2}} \bigg(\int_{B_R} \big|Du(x) - Dv_0(x)\big|^{p_2}\,dx\bigg)^{1/p_2} \\
&\quad + cH(R)F(R)^{\frac{p_2\beta}{1-p_2}} + c\big[F(R)\big]^{p_2\beta} \int_{B_R} \big(1 + \big|Du(x)\big|^{p_2}\big)\,dx \\
&\leqslant c\bigg[\bigg(\frac{\rho}{R}\bigg)^n + \varepsilon\bigg] \int_{B_R} \big(1 + \big|Du(x)\big|^{p_2}\big)\,dx + \bar{c}R^\lambda + cH(R) + cH(R)\big[F(R)\big]^{\frac{p_2\beta}{1-p_2}} + c\big[F(R)\big]^{p_2\beta} \int_{B_R} \big(1 + \big|Du(x)\big|^{p_2}\big)\,dx,
\end{aligned}$$

for any $0 < \rho < R$, where $c \equiv c(n, \gamma_1, \gamma_2, L, M, \lambda, \gamma)$ and $\bar{c} \equiv \bar{c}(n, \gamma_1, \gamma_2, L, M, \varepsilon, \lambda, \gamma)$. Now we choose $\beta \equiv \beta(\gamma_1) > 0$ such that

$$\beta < \frac{\gamma_1 - 1}{\gamma_1^2} \leqslant \frac{p_2 - 1}{p_2^2}.$$

With this choice of β we deduce

$$H(R)\big[F(R)\big]^{\frac{p_2\beta}{1-p_2}} \leqslant cF(R)^{1-1/p_2} \bigg[\int_{B_{2R}} \big(1 + \big|Du(x)\big|^{p_2}\big)\,dx + R^\lambda\bigg].$$

Therefore, combining the previous facts, we easily get

$$\int_{B_\rho} \big|Dv_0(x)\big|^{p_2}\,dx \leqslant c\bigg[\bigg(\frac{\rho}{R}\bigg)^n + \varepsilon\bigg] \int_{B_R} \big(1 + \big|Du(x)\big|^{p_2}\big)\,dx + c\big[F(R)\big]^{1-1/p_2} \int_{B_{2R}} \big(1 + \big|Du(x)\big|^{p_2}\big)\,dx + \bar{c}R^\lambda. \tag{5.17}$$

Using once more (5.15)(i), we end up with

$$\begin{aligned}
\int_{B_\rho} \big|Du(x)\big|^{p_2}\,dx &\leqslant c\int_{B_\rho} \big|Dv_0(x)\big|^{p_2}\,dx + c\int_{B_\rho} \big|Du(x) - Dv_0(x)\big|^{p_2}\,dx \\
&\leqslant c\bigg[\bigg(\frac{\rho}{R}\bigg)^n + \big[F(R)\big]^{1-1/p_2} + \varepsilon\bigg] \int_{B_{2R}} \big(1 + \big|Du(x)\big|^{p_2}\big)\,dx + \bar{c}R^\lambda,
\end{aligned} \tag{5.18}$$

for any $0 < \rho < R$, where $c \equiv c(n, \gamma_1, \gamma_2, L, M, \lambda, \gamma, r)$ and $\bar{c} \equiv \bar{c}(n, \gamma_1, \gamma_2, L, M, \varepsilon, \lambda, \gamma, r)$.

Let B_{R_0} be a ball such that $B_{R_0} \subset \mathcal{O}$. Then estimate (5.18) holds for any radii $0 < \rho \leqslant R \leqslant R_0$. Let $\varepsilon_0 \equiv \varepsilon_0(n, M, L, \gamma_1, \gamma_2, \lambda, \gamma)$ be the quantity provided by Lemma 3.5. We choose $\varepsilon = \varepsilon_0/2$ and this fixes the dependencies in (5.18) of the constant $\bar{c} \equiv \bar{c}(n, \gamma_1, \gamma_2, L, M, \lambda, \gamma)$. Then by our assumption (2.9) we can find a radius $R_1 > 0$ so small that $[F(R)]^{1-1/p_2} < \varepsilon_0/2$ and therefore $[F(R)]^{1-1/p_2} + \varepsilon < \varepsilon_0$ for any $0 < R \leqslant R_1$ and thus we have $R_1 \equiv R_1(n, \gamma_1, \gamma_2, L, M, \omega_1, \omega_2, \lambda, \gamma)$. Now Lemma 3.5 yields

$$\int_{B_\rho} \left|Du(x)\right|^{p_2} dx \leqslant c\rho^\lambda,$$

with $c \equiv c(n, M, L, \gamma_1, \gamma_2, \lambda, \gamma)$, whenever $0 < \rho < R_1$. Since we have $\gamma_1 \leqslant p_2 \leqslant \gamma_2$, we deduce by standard covering argument that

$$Du \in L^{\gamma_1,\lambda}_{\mathrm{loc}}(\Omega),$$

and thus by Poincaré's inequality we conclude that $u \in \mathcal{C}^{0,\alpha}_{\mathrm{loc}}(\Omega)$ with $\alpha = 1 - \frac{n-\lambda}{\gamma_1}$. This finishes the proof.

6. Proof of Theorem 2.10

Step 1 *(Localization and freezing).* We start by adopting the same localization argument as in the proof of Theorems 2.8 and 2.9. Again we assume that the $p(x)$ energy of u is bounded, i.e. (5.4). Let δ_1 be the higher integrability exponent of Lemma 3.2 and δ_2 the exponent coming from the up-to-the-boundary Proposition 3.4. We define $\delta := \min\{\delta_1, \delta_2\}$ and let R_M be a radius so small that $\omega_1(8R_M) + \omega_2(8R_M) \leqslant \delta/4$. From now on let $\mathcal{O}$ be an open set whose diameter does not exceed R_M and let the exponents p_1 and p_2 be defined as in (4.2).

Let u be a local minimizer of the functional (1.1) in K. As in the proof of Theorem 2.8, we define

$$\mathcal{G}_0(v, B_R) := \int_{B_R} f\big(x_0, (u)_R, Dv(x)\big)\, dx =: \int_{B_R} \bar{g}\big(Dv(x)\big)\, dx, \tag{6.1}$$

and let $v \in u + W^{1,p_2}_0(B_R)$ be the unique solution of the problem

$$\min\big\{\mathcal{G}_0(w, B_R):\ w \in u + W^{1,p_2}_0(B_R), w \geqslant \psi\big\}. \tag{6.2}$$

We set $A(\eta) := D\bar{g}(\eta)$. As $f \in \mathcal{C}^2$, then $\bar{g} \in \mathcal{C}^2$ and it satisfies the conditions (H6), (H7) and (3.10) with exponent $p = p_2$ while the linear and continuous operator A fulfills (3.8) and (3.9) with exponent $p = p_2$. We also introduce $w \in v + W^{1,p_2}_0(B_R)$ to be the solution of the equation

$$\int_{B_R} \big\langle A\big(Dw(x)\big), D\varphi(x)\big\rangle dx = \int_{B_R} \big\langle A\big(D\psi(x)\big), D\varphi(x)\big\rangle dx \quad \text{for all } \varphi \in W^{1,p_2}_0(B_R). \tag{6.3}$$

Then, by the maximum principle we obtain that $w \geqslant \psi$ in B_R, since $v \geqslant \psi$ on ∂B_R. Since $v - w \in W^{1,p_2}_0(B_R)$ and $w \geqslant v$ in B_R, we conclude by the minimizing property of v:

$$\int_{B_R} \big\langle A\big(Dv(x)\big), Dv(x) - Dw(x)\big\rangle dx \leqslant 0. \tag{6.4}$$

At this point let z be the solution of the following minimum problem:

$$\min\big\{\mathcal{G}_0(z, B_R):\ z \in u + W^{1,p_2}_0(B_R)\big\}, \tag{6.5}$$

where $\mathcal{G}_0$ has been introduced in (6.1). For the seek of clearness we advise the reader to the fact that w and z are "free" minimizers, whereas u and v are minimizers in the appropriate obstacle classes. It is clear that z satisfies

$$\int_{B_R} \big\langle A\big(Dz(x)\big), D\varphi(x)\big\rangle dx = 0 \quad \text{for all } \varphi \in W^{1,p_2}_0(B_R);$$

moreover $z = w$ on ∂B_R, so for example

$$\int_{B_R} \big\langle A\big(Dz(x)\big), Dw(x) - Dz(x)\big\rangle dx = 0. \tag{6.6}$$

Step 2 *(Reference estimates).* We prove Theorem 2.10 by comparison to the minimizer z of the frozen problem in the whole class $u + W_0^{1,p_2}(B_R)$ which can be shown to fulfill a suitable reference estimate. Additionally we need to show a comparison estimate between z and the original minimizer u which is established via some comparison steps between. First we start with a reference estimate for z, then we compare z and w, after that w and v. Finally we compare v and u. Note that all comparisons between the functions v, w and z can be cited from [15], since these functions are solutions or minimizers, respectively, of suitable frozen problems with constant exponent p_2. Therefore we shorten these steps, only citing the results and the structure conditions needed, referring the reader to [15, Proof of Theorem 2.11] for a more detailed discussion.

Using the estimates (2.4) and (2.5) in [28] we deduce the reference estimate

$$\fint_{B_\rho} \left|Dz(x) - (Dz)_\rho\right|^{p_2} dx \leqslant c\left(\frac{\rho}{R}\right)^{\beta p_2} \fint_{B_R} \left(1 + \left|Dz(x)\right|^{p_2}\right) dx, \tag{6.7}$$

where $c > 0$, $0 < \beta < 1$ and both c and β depend only on γ_1, γ_2, L. On the other hand a classical result, proved in Theorem 2.2 in [17] gives us

$$\int_{B_\rho} \left|Dz(x)\right|^{p_2} dx \leqslant c\left(\frac{\rho}{R}\right)^{n} \int_{B_R} \left(1 + \left|Dz(x)\right|^{p_2}\right) dx, \tag{6.8}$$

with $c \equiv c(\gamma_1, \gamma_2, L)$.

Step 3 *(Comparison estimates).* Exploiting the fact that by Theorem 2.6 we have

$$D\psi \in \mathcal{L}^{\gamma_1,\lambda}_{\mathrm{loc}}(\Omega) \quad \Rightarrow \quad D\psi \in C^{0,\alpha}(\overline{B_R}),$$

for any $B_R \Subset \Omega$, where $\alpha = \frac{\lambda - n}{\gamma_1}$, and since our results are local in nature, we may assume without loss of generality that for any $p_2 > 1$ we have

$$\left|A\left(D\psi(x)\right) - A\left(D\psi(y)\right)\right| \leqslant c|x-y|^{\alpha(p_2-1)}, \tag{6.9}$$

for all $x, y \in \Omega$. This allows us to deduce, estimating exactly as in [15, pp. 166, 167], that

$$\int_{B_R} \left|Dw(x) - Dz(x)\right|^{p_2} dx \leqslant c(L) R^{\frac{\alpha(p_2-1)}{2}} \int_{B_R} \left(1 + \left|Dw(x)\right|^{p_2}\right) dx. \tag{6.10}$$

On the other hand by the minimality of z we immediately deduce

$$\int_{B_R} \left|Dz(x)\right|^{p_2} dx \leqslant c(L) \int_{B_R} \left(1 + \left|Dw(x)\right|^{p_2}\right) dx. \tag{6.11}$$

Moreover, using (3.8), (3.9) and (6.3), we deduce, following line by line the estimates in [15, pp. 167, 168], the estimate

$$\int_{B_R} \left|Dw(x)\right|^{p_2} dx \leqslant c \int_{B_R} \left(\left|Dv(x)\right|^{p_2} + 1\right) dx, \tag{6.12}$$

with $c \equiv c(L, \gamma_1, \gamma_2, \alpha)$. This, together with (6.11) and the minimality of v in K, yields

$$\int_{B_R} \left|Dz(x)\right|^{p_2} dx \leqslant c \int_{B_R} \left(1 + \left|Du(x)\right|^{p_2}\right) dx, \tag{6.13}$$

with $c \equiv c(L, \gamma_1, \gamma_2, \alpha)$. The comparison between v and w can be established in an analogue way to the one to establish (6.10), obtaining

$$\int_{B_R} \left|Dv(x) - Dw(x)\right|^{p_2} dx \leqslant c R^{\frac{\alpha(p_2-1)}{2}} \int_{B_R} \left(\left|Dv(x)\right|^{p_2} + 1\right) dx, \tag{6.14}$$

with $c \equiv c(L)$. Now we compare u and v. First of all we discover that by the fact that $D\psi \in C^{0,\alpha}(\overline{B_R})$ for any $B_R \Subset \Omega$, since we only prove local results, we may assume that $D\psi \in L^\infty(\Omega)$ and therefore by Proposition 2.2 in [21] that $D\psi \in L^{p,n}(\Omega)$ for any $p > 1$. This allows us to deduce the estimate

$$\int_{B_R} \left|D\psi(x)\right|^{p_2} dx \leqslant cR^n. \tag{6.15}$$

Going through the proof of Lemma 3.2 and using estimate (6.15) one can easily see that we have higher integrability for u in the following sense:

$$\left(\fint_{B_{R/2}} \left|Du(x)\right|^{p(x)(1+\delta)} dx \right)^{1/(1+\delta)} \leqslant c_0 \fint_{B_R} \left(\left|Du(x)\right|^{p(x)} + 1\right) dx, \tag{6.16}$$

with $c_0 \equiv c_0(n, \gamma_1, \gamma_2, L, M)$, M being the bound for the $p(x)$ energy of u. Proposition 3.4 together with a similar argument allows us to conclude

$$\left(\fint_{B_R} \left|Dv(x)\right|^{p_2(1+\tilde{\varepsilon})} dx \right)^{1/p_2(1+\tilde{\varepsilon})} \leqslant c\left(\fint_{B_R} \left|Dv(x)\right|^{p_2} dx \right)^{1/p_2} + c\left[\fint_{B_R} \left(1 + \left|Du(x)\right|^{p_2(1+\delta/4)}\right) dx \right]^{1/p_2(1+\delta/4)}, \tag{6.17}$$

for any $\tilde{\varepsilon} \in (0, \delta/4)$, with $c \equiv c(n, \gamma_1, \gamma_2, L, M)$. At this point, working exactly as in the proof of Theorem 2.9 but using this time (6.16) and (6.17) instead of (3.2) and (3.11) respectively, we obtain

$$\mathcal{G}_0(u) - \mathcal{G}_0(v) \leqslant c\left(\omega_1(R)\log\left(\frac{1}{R}\right) + \omega_2\left(R^{\tilde{m}}\right)\right)\left[\int_{B_{2R}} \left(\left|Du(x)\right|^{p_2} + 1\right) dx + R^n \right],$$

which in turn entails, using assumption (2.11) and recalling the definition of $\tilde{m}$ given in Section 5

$$\mathcal{G}_0(u) - \mathcal{G}_0(v) \leqslant cR^{\zeta}\left[\int_{B_{2R}} \left(\left|Du(x)\right|^{p_2} + 1\right) dx \right],$$

where $\zeta \equiv \zeta(n, \gamma_1, \gamma_2, \delta, \gamma, \lambda, \varsigma)$ and $c \equiv c(n, \gamma_1, \gamma_2, L, M, \gamma)$, recalling that ς denotes the Hölder exponent appearing in (2.11). Since the integrand is of class $\mathcal{C}^2$, we can exploit (3.10), arguing exactly as in [2, pp. 131, 137, 138] to conclude

$$\int_{B_R} \left|Du(x) - Dv(x)\right|^{p_2} dx \leqslant cR^{\zeta/2}\left[\int_{B_{2R}} \left(\left|Du(x)\right|^{p_2} + 1\right) dx \right], \tag{6.18}$$

with $c \equiv c(n, \gamma_1, \gamma_2, L, M, \gamma)$. Thus summing up, taking together the estimates (6.10), (6.12), (6.14) and (6.18), additionally setting

$$\mathcal{M} := \min\left\{ \frac{\alpha(p_2 - 1)}{2}, \frac{\zeta}{2} \right\}$$

we conclude, using the minimality of v in K,

$$\begin{aligned} \int_{B_R} \left|Dz(x) - Du(x)\right|^{p_2} dx &\leqslant c(\gamma_2)\left[\int_{B_R} \left|Dz(x) - Dw(x)\right|^{p_2} dx + \int_{B_R} \left|Dw(x) - Dv(x)\right|^{p_2} dx + \int_{B_R} \left|Dv(x) - Du(x)\right|^{p_2} dx \right] \\ &\leqslant cR^{\mathcal{M}} \int_{B_{2R}} \left(1 + \left|Du(x)\right|^{p_2}\right) dx, \end{aligned} \tag{6.19}$$

with $c \equiv c(n, \gamma_1, \gamma_2, L, M, \gamma, \alpha)$.

Step 4 *(Conclusion).* Combining this comparison estimate with the reference estimate (6.7) and using (6.13) we deduce for any $0 < \rho < R/2 < R_0/2$,

$$\begin{aligned} \int_{B_\rho} \left|Du(x) - (Du)_\rho\right|^{p_2} dx &\leqslant c(\gamma_2)\left[\int_{B_\rho} \left|Dz(x) - (Dz)_\rho\right|^{p_2} dx + \int_{B_\rho} \left|Dz(x) - Du(x)\right|^{p_2} dx \right] \\ &\leqslant c\left(\frac{\rho}{R}\right)^{\beta p_2 + n} \int_{B_R} \left(1 + \left|Dz(x)\right|^{p_2}\right) dx + cR^{\mathcal{M}} \int_{B_{2R}} \left(1 + \left|Du(x)\right|^{p_2}\right) dx \\ &\leqslant c\left[\left(\frac{\rho}{R}\right)^{\beta p_2 + n} + R^{\mathcal{M}}\right] \int_{B_{2R}} \left(\left|Du(x)\right|^{p_2} + 1\right) dx, \end{aligned} \tag{6.20}$$

with a constant c depending on $n, \gamma_1, \gamma_2, L, M, \gamma$ and α. On the other hand, using (6.8), (6.19) and (6.13), we get

$$\begin{aligned}\int_{B_\rho} \left|Du(x)\right|^{p_2} dx &\leqslant c(\gamma_2)\left[\int_{B_\rho} \left|Dz(x)\right|^{p_2} dx + \int_{B_R} \left|Dz(x) - Du(x)\right|^{p_2} dx\right] \\ &\leqslant c\left(\frac{\rho}{R}\right)^n \int_{B_R} \left(1 + \left|Dz(x)\right|^{p_2}\right) dx + cR^{\mathcal{M}} \int_{B_{2R}} \left(\left|Du(x)\right|^{p_2} + 1\right) dx \\ &\leqslant c\left[\left(\frac{\rho}{R}\right)^n + R^{\mathcal{M}}\right] \int_{B_{2R}} \left|Du(x)\right|^{p_2} dx + cR^n,\end{aligned}$$

with a constant $c \equiv c(n, \gamma_1, \gamma_2, L, M, \gamma, \alpha)$. By the above estimate we conclude that we have in particular for $\phi(\rho) := \int_{B_\rho} |Du|^{p_2}\, dx$ the estimate

$$\phi(\rho) \leqslant \bar{c}\left[\left(\frac{\rho}{R}\right)^n + R^{\mathcal{M}}\right]\phi(2R) + cR^{n-\tau},$$

for any $\tau \in (0, 1)$ and for all $0 < \rho \leqslant R \leqslant R_0$.

Now let $\varepsilon_0 \equiv \varepsilon_0(\bar{c}, n, \tau)$ the quantity of Lemma 3.5. Then we choose $R_{00} \leqslant R_0$ so small that $R^{\mathcal{M}} < \varepsilon_0$ for any $0 < R \leqslant R_{00}$. This yields $R_{00} \equiv R_{00}(n, L, \gamma_1, \gamma_2, M, \gamma, \alpha)$. Then Lemma 3.5, together with the p_2 energy bound of u yields

$$\phi(\rho) \leqslant c\rho^{n-\tau},$$

for any $0 < \rho < R_{00}$, with $c \equiv c(n, M, R_0, \tau, L, \gamma_1, \gamma_2, \alpha, \gamma)$. Since estimate (6.20) holds for any $0 < \rho \leqslant R \leqslant R_{00}$ we apply this estimate for $\rho \equiv \frac{1}{2}R^{1+\theta}$ with $\theta \equiv \frac{\mathcal{M}}{n+p_2\beta}$. Next we choose $\tau \equiv \frac{1}{2}\frac{p_2\mathcal{M}\beta}{n+p_2\beta}$ which fixes the dependency of the above constant. The choice of the quantities then yields

$$\int_{B_\rho} \left|Du(x) - (Du)_\rho\right|^{p_2} dx \leqslant c(L, \gamma_1, \gamma_2, \alpha)\rho^{\tilde{\lambda}}, \tag{6.21}$$

where

$$\tilde{\lambda} := n + \frac{p_2\beta\mathcal{M}}{2(n + p_2\beta + \mathcal{M})}.$$

But the choice of R was arbitrary, so without loss of generality we may assume that (6.21) holds for all $0 < \rho \leqslant R_{00}$. Now we would like to conclude by Theorem 2.6; thus we have to make sure that $\tilde{\lambda} < n + \gamma_1$. If $\beta < (2\gamma_1)/\gamma_2$, this is true, otherwise we can choose $\mathcal{M}$ sufficiently small (this is not restrictive). Thus Theorem 2.6 gives that $Du \in C^{0,\tilde{\alpha}}_{\mathrm{loc}}(\Omega)$ with $\tilde{\alpha} = 1 - \frac{n-\tilde{\lambda}}{\gamma_1}$. This yields the thesis.

Acknowledgments

The first author wishes to thank the kind hospitality of the Mathematisches Institut of the Friedrich-Alexander-Universität Erlangen-Nürnberg, together with the nice and friendly atmosphere there. He wishes also to thank the kind hospitality of the Weierstrass Institute in Berlin during her stay as WIAS Fellow in 2007. The authors wish also to acknowledge F. Duzaar and G. Mingione for useful discussions.

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